A METRIC ON THE SET OF CONNECTED SIMPLE GRAPHS OF GIVEN ORDER

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Abstract

We introduce a sequence d_n^ℓ $(\ell = 1, 2, \ldots)$ of functions on $\mathcal{G}_n \times \mathcal{G}_n$, where $\mathcal{G}_n$ is the set of all simple, connected, undirected graphs of order n up to isomorphism. We show that when $\ell = 1$ or $\ell \geqslant n-1$, d_n^ℓ is a metric on $\mathcal{G}_n$. While (G_n, d_n^1) is a totally disconnected metric space that embodies the classical notion of graph isomorphism, (G_n, d_n^ℓ) is a connected metric space whenever $\ell \geqslant n-1$. In this paper, we investigate some properties and the relationship between these two spaces. This work was motivated by the problem of virtual path layout in high-speed computer networks, which concerns embedding a specified virtual network into the given physical network in a way that makes optimal use of the physical network resources.

1 Introduction

We begin with an informal, brief description of the concepts and motivation behind this work.

Given two connected, simple graphs H, K of order n, begin by considering a one-to-one mapping $\phi : V[K] \to V[H]$. Upon fixing ϕ, to each edge $e = uv \in E[K]$ we associate a path p_e between $\phi(u)$ and $\phi(v)$ in H. We view ϕ together with the chosen set of paths $Q = \{p_e | e \in E[K]\}$ as a topological embedding ϕ' of K into H. If all the paths in Q are of length $\leqslant \ell$, then ϕ' is called an ℓ-topological embedding of K into H. Given ϕ', each edge e in $E[H]$ is traversed some number of times by the paths in Q. We call this number "the congestion on e" with respect to the embedding ϕ'—it is, in some sense, the size of the fibre over edge e under ϕ'. For a given embedding ϕ' of K into H, we will be interested in the maximum congestion on the edges of $E[H]$. In general, there are many (or none) ℓ-topological embeddings of K into H, and we shall be seeking those embeddings which minimize the maximum congestion. The logarithm of this minimal value will be later defined as the ℓ-embedding thickness of K in H, denoted $e_n^\ell(H, K)$.

The situation above is a reasonable abstract model of the practical problem of virtual path layout in computer networks [1, 3, 5]: Given a physical computer network H consisting of switches $V[H]$ connected by fiber-optic cables $E[H]$ of fixed capacity, we desire

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to implement a specific virtual computer network K using the physical hardware of H. This amounts to finding a topological embedding of K into H. To minimize transmission delay, the links of the virtual network $E[K]$ must be implemented as network connections (i.e. paths) of length $\leqslant \ell$ in the physical network H. Since these connections utilize the bandwidth of links in the physical network, it is desirable to use an embedding which minimizes the maximum congestion over the physical links.

Just as we considered the topological embeddings of K into H to determine $e_n^{\ell}(H, K)$, we now consider embeddings of H into K to determine $e_n^{\ell}(K, H)$. The sum of these two functions will be defined as the ℓ-distance between H and K, denoted $d_n^{\ell}(H, K)$. In this paper, we will analyze the properties of e_n^{ℓ} and d_n^{ℓ}. In particular, we will show that when $\ell = 1$ and $\ell \geqslant n-1$, d_n^{ℓ} equips the set of simple connected graphs of order n with a metric structure.

In an earlier paper, G. Chartrand, G. Kubicki and M. Schultz [2] presented a different metric on the set of graphs of order n, which they termed ϕ-distance. Their measure was based on embeddings of graphs which minimize the absolute distortion of pairwise distances between vertices. In contrast, the metric proposed here is based on consideration of embeddings between graphs which minimize the maximum size fibre over any edge. For us, the study of such embeddings originated in the practical problem of virtual path layout on computer networks.

The previous descriptions will now be formulated precisely.

Given an undirected graph $G = (V, E)$, recall that a **path** of length l in G is a sequence of $l+1$ distinct vertices $p = (v_0, v_1, \cdots, v_l)$, where $v_i \in V$ for $i = 0, \ldots, l$, and $v_j v_{j+1} \in E$ for $j = 0, \ldots, l-1$, see [4]. We define the **boundary** of p as $\partial p = \{v_0, v_l\}$, and will denote the set of all unordered pairs of vertices in G as $V \times V$.

`Definition 1.1.` Define $\mathcal{P}_G^l$ to be the set of all paths in G of length at most l, and take $\mathcal{P}_G = \bigcup_{l=1}^{\infty} \mathcal{P}_G^l$.

`Definition 1.2.` To each set of paths $Q \subseteq \mathcal{P}_G$, associate the **path multi-graph** $Q^{\circ} = (V, E_Q)$, where $uv \in E_Q \Leftrightarrow \exists p \in Q$ such that $\partial p = \{u, v\}$.

`Definition 1.3.` For $p \in \mathcal{P}_G$ and $e \in E$, let $m(p, e)$ denote the multiplicity of e in p. For $Q \subseteq \mathcal{P}_G$ define the congestion of Q at e as $m(Q, e) = \sum_{p \in Q} m(p, e)$. Finally, given an undirected graph G, we define the **congestion** of Q on G as $\tau_G(Q) = \max_{e \in E} m(Q, e)$.

`Definition 1.4.` Let $\mathcal{G}_n$ be the set of all simple, connected, undirected graphs (up to isomorphism) on n vertices. For each positive integer ℓ and graphs H, K in $\mathcal{G}_n$, we define the **ℓ-embedding thickness** of K in H, denoted $e_n^{\ell}(H, K)$, as follows: If there exists a set $Q \subseteq \mathcal{P}_H^{\ell}$ such that $Q^{\circ} \simeq K$ then consider Q for which $\tau_H(Q) = 2^x$ is minimal, and set $e_n^{\ell}(H, K) = x$, otherwise set $e_n^{\ell}(H, K) = \infty$. We will write $H \succ_{\ell} K$ if $e_n^{\ell}(H, K) = 0$.

The **embedding thickness** of K in H, denoted by $e_n^{*}(H, K)$, is obtained as above except that $Q \subseteq \mathcal{P}_H$; that is, Q is a set of paths of arbitrary lengths. We will write $H \succ_* K$ when $e_n^*(H,K) = 0$.

`Remark 1.5.` Note that $e_n^1(H,K) = 0$ implies that there is a set of edge-disjoint paths $Q \subseteq \mathcal{P}_H^1 = E[H]$ such that $Q^\circ \simeq K$, and so $|E[K]| \leqslant |E[H]|$.

`Definition 1.6.` For any graphs H, K in $\mathcal{G}_n$, we define their **ℓ-distance** and **distance**, respectively, as follows:

$$\begin{array}{rcl} d_n^\ell(H,K) & = & e_n^\ell(H,K) + e_n^\ell(K,H) \\ d_n^*(H,K) & = & e_n^*(H,K) + e_n^*(K,H). \end{array}$$

Note that $d_n^\ell(H,K)$ may be infinity—for example, when $\ell = 1$ and K is a proper connected spanning subgraph of H.

2 Results

Remark 1.5 implies the following lemma.

`Lemma 2.1.` *$H \succ_1 K \Leftrightarrow K$ is a connected spanning subgraph of H.*

Since $\mathcal{P}_H^\ell \subseteq \mathcal{P}_H^{\ell'}$ for $1 \leqslant \ell \leqslant \ell'$, the next lemma follows from definition 1.4.

`Lemma 2.2.` *$\forall l, l', 1 \leqslant \ell < \ell'$ implies $\forall H, K \in \mathcal{G}_n$, $e_n^{\ell'}(H,K) \leqslant e_n^\ell(H,K)$.*

As a corollary, the relations $\succ_\ell$, $\ell \in \mathbb{Z}^+$ form an ascending sequence of binary relations on $\mathcal{G}_n$.

`Corollary 2.3.` *$\forall l, l', 1 \leqslant \ell < \ell'$ implies $\forall H, K \in \mathcal{G}_n$, $H \succ_\ell K \Rightarrow H \succ_{\ell'} K$.*

The next proposition shows the functions $\{e_n^\ell\}_{l \in \mathbb{Z}^+}$ satisfy a graded triangle inequality.

`Proposition 2.4.` *For any $\ell_1, \ell_2 \in \mathbb{Z}^+$, and any $G, H, K \in \mathcal{G}_n$,*

$$e_n^{\ell_1 \ell_2}(G,K) \leqslant e_n^{\ell_1}(G,H) + e_n^{\ell_2}(H,K)$$

Proof. Without loss of generality, we assume that the terms on the right-hand side are both finite, since otherwise, there is nothing to prove.

Suppose that (i) $e_n^{\ell_1}(G,H) = x$ and (ii) $e_n^{\ell_2}(H,K) = y$. Then this implies that (i) $\exists Q \subseteq \mathcal{P}_G^{\ell_1}$ such that $Q^\circ \simeq H$, and $\tau_G(Q) = 2^x$, and (ii) $\exists R \subseteq \mathcal{P}_H^{\ell_2}$ such that $R^\circ \simeq K$ and $\tau_H(R) = 2^y$.

Since every edge of K is mapped to a path of length at most ℓ_2 in H, and every edge of H is mapped to a path of length at most ℓ_1 in G, it follows that the composite map sends every edge of K to a walk of length at most $\ell_1 \ell_2$ in G. Such a walk can then be reduced to a path by eliminating loops.

Further, since the congestion of Q in G is 2^x and the congestion of R in H is 2^y, it follows that the paths in G corresponding to the edges of K (as defined above) have congestion at most $2^x 2^y$. Hence $e_n^{\ell_1 \ell_2}(G, K) \leqslant x + y$. $\square$

The well-known fact that "If H is a subgraph of K, and K is a subgraph of H then $H \simeq K$" can be extended to the relations $\succ_\ell$, $\forall \ell \in \mathbb{Z}^+$, as the next result shows.

Theorem 2.5. *$\forall \ell \in \mathbb{Z}^+, \forall H, K \in \mathcal{G}_n$, $[H \succ_\ell K$ and $K \succ_\ell H$ implies $H \simeq K]$*

Proof. Suppose $H, K \in \mathcal{G}_n$ such that (i) $e_n^\ell(H, K) = 0$ and (ii) $e_n^\ell(K, H) = 0$. Then this implies that (i) $\exists Q \subseteq \mathcal{P}_H^\ell$ such that $Q^\circ \simeq K$ and $\tau_H(Q) = 1$, and (ii) $\exists R \subseteq \mathcal{P}_K^\ell$ such that $R^\circ \simeq H$ and $\tau_K(R) = 1$.

We show first that $|E[H]| = |E[K]|$. Suppose not, for a contradiction. Then, WLOG $|E[H]| > |E[K]|$. But $H \simeq R^\circ$, $R \subseteq \mathcal{P}_K^\ell$, implying that $|R| > |E[K]|$. By pigeonhole argument, $\exists e \in E[K]$ such that $\exists r_1, r_2 \in R$ distinct, both traversing the edge e, which contradicts the assumption that $\tau_K(R) = 1$. So $|E[H]| = |E[K]|$.

Note that $|R| = |E[H]| = |E[K]| = |Q|$. Each $q \in Q$ is a path in H; since $\tau_H(Q) = 1$ and $|E[H]| = |Q|$, this implies that each $q \in Q$ is a path of length 1 in H. Likewise, each $r \in R$ is a path in K; since $\tau_K(R) = 1$ and $|E[K]| = |R|$, this implies that each $r \in R$ must be a path of length 1 in K. It follows from lemma 2.1 that H is a connected spanning subgraph of K, and K is a connected spanning subgraph of H; hence $H \simeq K$. $\square$

Thus we have obtained the following the result: $\forall \ell \in \mathbb{Z}^+, \forall H, K \in \mathcal{G}_n$, $d_n^\ell(H, K) = 0 \Leftrightarrow H \simeq K$.

Since every set of paths $Q \subseteq \mathcal{P}_G^\ell$ is also a subset of $\mathcal{P}_G^{n-1}$, we obtain the next lemma.

Lemma 2.6. *$\forall \ell, \ell' \geqslant n - 1$, $e_n^\ell \equiv e_n^{\ell'}$, that is the relations $\succ_\ell$ are constant.*

The previous lemma and proposition 2.4 yield the following result.

Corollary 2.7. *If $\ell = 1$ or $\ell \geqslant n - 1$, then e_n^ℓ satisfies the triangle inequality.*

As a corollary we extend the well-known assertion that "If G is a subgraph of H, and H is a subgraph of K, then G is a subgraph of K" to the relations $\succ_\ell$, when $\ell = 1$ or $\ell \geqslant n - 1$.

Corollary 2.8. *If $\ell = 1$ or $\ell \geqslant n - 1$, then $\forall\, G, H, K \in \mathcal{G}_n$, $G. \succ_\ell H$ and $H \succ_\ell K$ implies $G \succ_\ell K$*

Proof. By definition, if $G \succ_\ell H$ and $H \succ_\ell K$ then $e_n^\ell(G,H) = e_n^\ell(H,K) = 0$. Then, by Corollary 2.7, $e_n^\ell(G,K) = 0$, so $G \succ_\ell K$. ☐

`Theorem 2.9.` *If $\ell = 1$ or $\ell \geqslant n-1$, $(\mathcal{G}_n, d_n^\ell)$ is a metric space.*

Proof. Clearly, d_n^ℓ is a symmetric function. By Corollary 2.7, e_n^ℓ satisfies the triangle inequality when $\ell = 1$ or $\ell \geqslant n-1$. Finally, reflexivity was shown in Theorem 2.5. ☐

3 Some Properties of $(\mathcal{G}_n, d_n^\ell)$

In lemma 2.1, we showed that the relation $\succ_1$ precisely captures the notion of a connected spanning subgraph. We now show that for n sufficiently large, the relation $\succ_*$ is strictly weaker than $\succ_1$.

The next lemma follows from definition 1.4 and lemma 2.1.

`Lemma 3.1.` $\forall H, K \in \mathcal{G}_n,\ H \succ_1 K \Rightarrow H \succ_* K$

Note that lemma 3.1 holds for $\succ_\ell$ (for all ℓ). However the converse of the above lemma is false.

`Lemma 3.2.` $\forall n \geqslant 5,\ \exists H, K \in \mathcal{G}_n$ *s.t.* $H \succ_* K$ *and* $H \not\succ_1 K$

Proof. Fix $n \geqslant 5$, and take graphs $H, K \in \mathcal{G}_n$ to be the graphs depicted in figure 1. Let $Q \subseteq \mathcal{P}_H$ be the set of paths in H chosen as follows:

$$Q \quad = \quad \{(a,b),\ (b,c),\ (b,d),\ (a,d,c),\ (d,e_1),\ (e_1,e_2),\ \ldots,\ (e_{n-3},e_{n-4})\}$$

Then $Q^\circ \simeq K$, and $\tau_H(Q) = 1$. This shows $e_n^*(H,K) = 0$, i.e. $H \succ_* K$. It is an easy exercise to check that K is not a subgraph of H, i.e. $H \not\succ_1 K$. ☐

`Lemma 3.3.` *Let $H = (V,E) \in \mathcal{G}_n$, and $e = uv \in (V \times V) \backslash E$. Let $H' = (V, E \cup e)$. Then $d_n^*(H,H') \leqslant 1$.*

Proof. $H' \succ_1 H$, so by lemma 3.1, $H' \succ_* H$, i.e. $e_n^*(H',H) = 0$. Let $Q = E \cup p$ where p is any path in H such that $\partial p = \{u,v\}$. Without loss of generality, we may take p to be non self-intersecting. Then $Q^\circ \simeq H'$ and $\tau_H(Q) \leqslant 2$. This shows that $e_n^*(H,H') \leqslant \log_2 2 = 1$. By definition, $d_n^*(H',H) = e_n^*(H',H) + e_n^*(H,H') \leqslant 1$. ☐

`Lemma 3.4.` $\forall H, K \in \mathcal{G}_n$, *$d_n^1(H,K)$ is 0 if $H = K$ and infinite otherwise.*

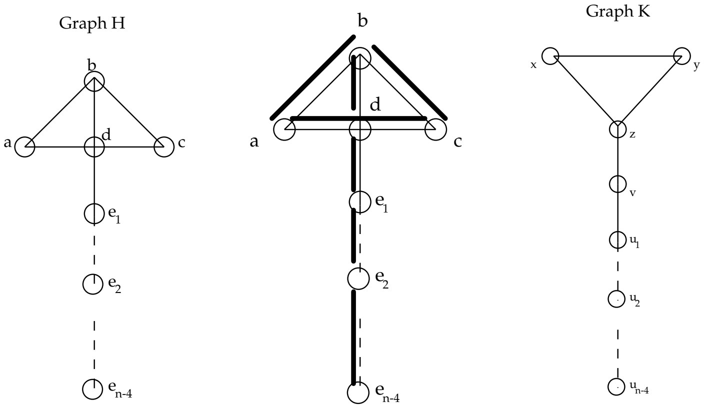

Figure 1: A set of paths certifying that $e_n^*(H,K) = 0$.

Proof. If $H = K$ then H is a subgraph of K and K is a subgraph of H, hence $e_n^1(H,K) = e_n^1(K,H) = 0$. It follows that $d_n^1(H,K) = 0$. If $H \neq K$, then either H is not a subgraph of K or K is not a subgraph of H. Suppose, WLOG that K is not a subgraph of H. Then, $\forall Q \subseteq \mathcal{P}_H^1$, $Q^\circ \not\simeq K$, so by definition, $e_n^1(H,K) = \infty$. It follows that $d_n^1(H,K) = \infty$. ☐

`Lemma 3.5.` *$\forall H, K \in \mathcal{G}_n$, $d_n^*(H,K)$ is finite, hence $(\mathcal{G}_n,\, d_n^*)$ is a connected metric space.*

Proof. One can transform H into K by a finite sequence of transformations:
$H = G_0 \rightsquigarrow G_1 \rightsquigarrow \ldots \rightsquigarrow G_m = K$, where for $i = 0, \ldots, m-1$, the transformation $G_i \rightsquigarrow G_{i+1}$ is the addition or deletion of a single edge. It is possible to construct such a transformation sequence by first sequentially adding edges to H until the complete graph K_n is attained, and then deleting edges as needed to arrive at the graph K. Clearly, all the intermediate graphs G_i are simple, connected, undirected graphs on n vertices, hence $\forall i = 0, \ldots, m, G_i \in \mathcal{G}_n$. By lemma 3.3, $d_n^*(G_i, G_{i+1}) \leqslant 1$, and theorem 2.9 asserts that d_n^* satisfies the triangle inequality; hence $d_n^*(H,K) \leqslant m < \infty$. ☐

`Definition 3.6.` Given a graph $G \in \mathcal{G}_n$, we define the ℓ-neighborhood of G as

$$\ell\text{-}nbd(G) = \{H \in \mathcal{G}_n \mid d_n^\ell(G,H) \text{ is finite. }\}$$

Since $l \geqslant n-1$ implies $d_n^\ell \equiv d_n^*$, we refer to ℓ-nbd as $*$-nbd whenever $l \geqslant n-1$. In this language, lemma 3.4 can be restated as $\forall G \in \mathcal{G}_n$, 1-$nbd(G) = \{G\}$, while lemma 3.5 is seen to assert that $\forall G \in \mathcal{G}_n$, $*$-$nbd(G) = \mathcal{G}_n$.

The next lemma characterizes ℓ-$nbd(G)$ when G is a complete graph K_n.

Lemma 3.7. $\ell\text{-}nbd(K_n) = \{H \in \mathcal{G}_n \mid Diameter(H) \leqslant \ell\}$

Proof. To see that $\ell\text{-}nbd(K_n) \subseteq \{H \in \mathcal{G}_n \mid Diameter(H) \leqslant \ell\}$: Suppose $H = (V, E) \in \ell\text{-}nbd(K_n)$. Then $\exists Q \subseteq \mathcal{P}_H^\ell$ such that $Q^\circ \simeq K_n$. It follows that $\forall u, v \in V$, $\exists q_{u,v} \in Q$ with $\partial q_{u,v} = \{u, v\}$. But $Q \subseteq \mathcal{P}_H^\ell$, so $q_{u,v}$ has length $\leqslant \ell$. This shows that $Diameter(H) \leqslant \ell$.

To see $\ell\text{-}nbd(K_n) \supseteq \{H \in \mathcal{G}_n \mid Diameter(H) \leqslant \ell\}$: Suppose $H = (V, E) \in \mathcal{G}_n$ and $Diameter(H) \leqslant \ell$. Let Q be the set of $n(n-1)$ shortest paths between all pairs of vertices $u, v \in V$. Clearly, $Q^\circ \simeq K_n$ and $Q \subseteq \mathcal{P}_H^{Diam(H)} \subseteq \mathcal{P}_H^\ell$. It follows that $e_n^\ell(H, K_n) \leqslant \tau_Q(H) < \infty$. Now H is a subgraph of K_n, so by lemmas 2.1 and 2.2, $e_n^\ell(K_n, H) = 0$. Since $d_n^\ell(K_n, H) = e_n^\ell(K_n, H) + e_n^\ell(H, K_n)$, it follows that $d_n^\ell(K_n, H) < \infty$, that is $H \in \ell\text{-}nbd(K_n)$. ☐

Given a map $\phi : (M_1, d_1) \to (M_2, d_2)$ between metric spaces, ϕ is said to have distortion bounded by $\lambda \geqslant 1$, if $\forall m, m' \in M_1$,

$$\frac{1}{\lambda} d_1(m, m') \leqslant d_2(\phi m, \phi m') \leqslant \lambda d_1(m, m').$$

ϕ is called an isometry if its distortion is bounded by 1. We are interested in low-distortion embeddings of $(\mathcal{G}_n, d_n^*)$ onto itself. In particular, we would like to know if $\mathcal{G}_n$ possesses any non-trivial isometries.

As a partial answer, we note that $\mathcal{G}_2$ has no non-trivial isometries, since it has only one element, namely the path graph P_2. On the other hand, $\mathcal{G}_3$ has a nontrivial isometry which exchanges the complete graph K_3 and the path graph P_3. For $n = 4$, there are six points in $\mathcal{G}_4$. The distances between these graphs is given in the table below, where P_4 is the path, C_4 is the cycle, C_4^* is the cycle with one chord, K_4 is the complete graph, S_4 is the star and S_4^* is the star with one chord. Using this table, one can verify that $\mathcal{G}_4$ has no non-trivial isometry.

	P_4	C_4	C_4^*	K_4	S_4	S_4^*
P_4	0	1	$\log_2 3$	2	2	$\log_2 3$
C_4	1	0	1	$\log_2 3$	2	2
C_4^*	$\log_2 3$	1	0	1	$\log_2 3$	1
K_4	2	$\log_2 3$	1	0	$\log_2 3$	$\log_2 3$
S_4	2	2	$\log_2 3$	$\log_2 3$	0	1
S_4^*	$\log_2 3$	2	1	$\log_2 3$	1	0

We conjecture that for $n > 4$, G_n does not possess any non-trivial isometries.

We are also interested in embeddings of $(\mathcal{G}_n, d_n^*)$ into $(\mathcal{G}_{n+1}, d_{n+1}^*)$. We begin by exhibiting one such embedding which does not increase the distance between graphs.

`Definition 3.8.` Let $\phi_n : (\mathcal{G}_n, d_n^*) \to (\mathcal{G}_{n+1}, d_{n+1}^*)$ be the map which sends a graph $G \in \mathcal{G}_n$ to the graph obtained by adding a new vertex g_{n+1} connected to every vertex of G.

`Lemma 3.9.` *ϕ_n is an embedding of $(\mathcal{G}_n, d_n^*)$ into $(\mathcal{G}_{n+1}, d_{n+1}^*)$.*

Proof. Suppose $\phi_n G$ is isomorphic to $\phi_n H$ via π, where $G, H \in \mathcal{G}_n$. If $\phi_n G$ has a unique vertex of degree n, then g_{n+1} is clearly mapped to h_{n+1} by π, and therefore $\pi(G) = H$. If $\phi_n G$ has several vertices $D \subset V[G]$ of degree n, then we may adjust the isomorphism π on D to obtain a new isomorphism π' for which $\pi' g_{n+1} = h_{n+1}$. Thus $\pi'(G) = H$. ☐

`Lemma 3.10.` *For all $H, K \in \mathcal{G}_n$, $d_{n+1}^*(\phi_n H, \phi_n K) \leqslant d_n^*(H, K)$.*

Proof. Suppose $e_n^*(G, H) = x$. Then there exists a set of paths $Q \subset \mathcal{P}_G$ such that $Q^o \cong H$ and $\tau_G(Q) = 2^x$. Take $Q^+ = Q \cup \{(v, s) \mid v \in V[G]\}$; then clearly $\tau_{\phi_n G}(Q^+) = 2^x$ and $(Q^+)^o \cong \phi_n H$. It follows that $e_{n+1}^*(\phi_n G, \phi_n H) \leqslant x = e_n^*(G, H)$. An analogous argument shows that $e_{n+1}^*(\phi_n H, \phi_n G) \leqslant e_n^*(H, G)$, and hence $d_{n+1}^*(\phi_n H, \phi_n K) \leqslant d_n^*(H, K)$. ☐

We give an example that shows that ϕ_4 has distortion greater than 1: Take $G = P_4$, the path on 4 vertices, and $H = C_4^*$, the cycle on 4 vertices with one chord. Since G is a subgraph of H, $e_4^*(H, G) = 0$. On the other hand, one may verify that $e_4^*(G, H) = \log_2 3$, and that $d_5^*(\phi_4 G, \phi_4 H) = 1$. Thus, the distortion of ϕ_4 is at least $\log_2 3$. This example may be generalized for $n > 4$.

We now pose the following open questions:

1. What is the precise distortion of the embedding ϕ_n?
2. Are there embeddings of $(\mathcal{G}_n, d^*_n)$ into $(\mathcal{G}_{n+1}, d^*_{n+1})$ whose distortion is lower than that of ϕ_n?

4 Remarks

Many interesting graph theoretic properties are defined in terms of the presence of characteristic subgraphs, i.e. "G has property P iff $G \succ_1 H$ for some $H \in \mathcal{H}$", where $\mathcal{H}$ is a given family of graphs. Graph properties that are defined in this manner can be naturally weakened to their $*$-analogues by declaring that "G has property $*P$ iff $G \succ_* H$ for some $H \in \mathcal{H}$". By lemma 3.1, the set of graphs with property P is a subset of the set of graphs with property $*P$; by lemma 3.2 this containment may be proper.

As a concrete example, $G \in \mathcal{G}_n$ is said to be Hamiltonian if $G \succ_1 C_n$, where C_n is the cycle of n vertices. Analogously, we say that $G \in \mathcal{G}_n$ is $*$-Hamiltonian if $G \succ_* C_n$. The reader may check that the property of being $*$-Hamiltonian is weaker than the property of being Hamiltonian; in particular, every Eulerian graph is also $*$-Hamiltonian.

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