A GRAPHIC GENERALIZATION OF ARITHMETIC

BILAL KHAN *, KIRAN R. BHUTANI †, AND DELARAM KAHROBAEI ‡

Abstract. In this paper, we extend the classical arithmetic defined over the set of natural numbers $\mathbb{N}$, to a set containing all finite directed connected multigraphs having a pair of distinct distinguished vertices. Specifically, we introduce a model $\mathcal{F}$ on the set of such graphs, and provide an interpretation of the language of arithmetic $\mathcal{L} = \{0, 1, \leqslant, +, \times\}$ inside $\mathcal{F}$. The resulting model exhibits the property that the standard model on $\mathbb{N}$ embeds in $\mathcal{F}$ as a submodel, with the directed path of length n playing the role of the standard integer n. We will compare the theory of the larger structure $\mathcal{F}$ with classical arithmetic statements that hold in $\mathbb{N}$. For example, we explore the extent to which $\mathcal{F}$ enjoys properties like the associativity and commutativity of $+$ and $\times$, distributivity, cancellation and order laws, and decomposition into irreducibles.

Key words. arithmetic, graphs

AMS subject classifications. 05C99, 11U10

1. Introduction. The *language of arithmetic* $\mathcal{L}$ consists of two 0-ary relations **0** and **1**, one binary relation $\leqslant$, and two ternary relations $+$ and $\times$. In this paper, we generalize classical arithmetic defined over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, to the set F of all *flow graphs*: finite directed connected multigraphs with a pair of distinguished vertices (designated the *source* and *terminal*) which are either distinct or else are the entire vertex set of a trivial edge-free graph. We give natural interpretation for $\mathcal{L}$ on the set F. To avoid confusion with the standard model of arithmetic, the corresponding operations in F are denoted with a circumscribed circle. The new model $\mathcal{F} = \langle F, ⓪, ①, \circledleq, \oplus, \otimes \rangle$ is a natural extension of the standard model $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant +, \times \rangle$. Specifically, we exhibit an embedding $i : \mathcal{N} \overset{i}{\hookrightarrow} \mathcal{F}$ satisfying:

$$\begin{aligned} i(0) &= ⓪, \\ i(1) &= ①, \\ \forall x, y \in \mathbb{N}, \quad x \leqslant y \Rightarrow i(x) &\circledleq i(y), \\ \forall x, y \in \mathbb{N}, \quad i(x + y) &= i(x) \oplus i(y), \\ \forall x, y \in \mathbb{N}, \quad i(x \times y) &= i(x) \otimes i(y). \end{aligned}$$

Objective: *Compare the theory* $Th(\mathcal{F}) = \{\phi \mid \mathcal{F} \models \phi\}$ *with* true arithmetic $TA = \{\phi \mid \mathcal{N} \models \phi\}$[1].

There have been other attempts to define algebraic and metric structures on the set of all graphs. The classical operations on graphs [5] (including extensive literature on graph products [4]) have yielded deep results and a profound mathematical theory. However, to date, these operations have not provided an interpretation of the

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[1]Following standard model theory, here ϕ is a first-order sentence in the language $\mathcal{L}$.

language of arithmetic on graphs. This paper presents results and open questions in this direction. In [1, 2, 3], the authors used graph embeddings to define a metric on the set of all simple connected graphs of a given order. This work differs from those investigations in that it considers an infinite collection of graphs in order to extend the standard model of arithmetic, and in doing so does not seek to establish a metric structure.

DEFINITION 1.1 (Flow graph). *A flow graph A is a triple (G_A, s_A, t_A), where G_A is a finite directed connected multigraph and $s_A, t_A \in V[G_A]$ are called the* source *and the* target *vertex of A, respectively. Either $s_A \neq t_A$, in which case A is called a* non-trivial *flow graph, or $s_A = t_A$, $|V[G_A]| = 1$, and $|E[G_A]| = 0$, in which case A is called* the trivial flow graph. *The set of all flow graphs that are either non-trivial or trivial is denoted F. Two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$ are considered isomorphic if there is a* flow graph isomorphism *between them, that is, a graph isomorphism $\phi : G_A \to G_B$ satisfying $\phi(s_A) = s_B$, $\phi(t_A) = t_B$.*

DEFINITION 1.2 (Graphical natural number). *We represent the natural number n as a directed chain of length n, having $n+1$ vertices. More formally, let P_n be a directed chain of length n (having $n+1$ vertices) where each vertex has in-degree $\leqslant 1$ and out-degree $\leqslant 1$. Denote by s_n, the unique vertex in P_n having in-degree 0, and let t_n be the unique vertex in P_n having out-degree 0. The flow graph $F_n = (P_n, s_n, t_n)$ is referred* the graphic natural number n. *Define the map $i : \mathcal{N} \to \mathcal{F}$ as*

$$i : n \mapsto F_n.$$

1.1. Addition. In Definition 1.1, we represented the natural number n by the flow graph F_n. It follows that we interpret the addition of two numbers n_1 and n_2 inside $\mathcal{F}$ as "concatenating" F_{n_1} with F_{n_2}. Consider, for example, the addition of 3 and 2 depicted in Figure 1.1.

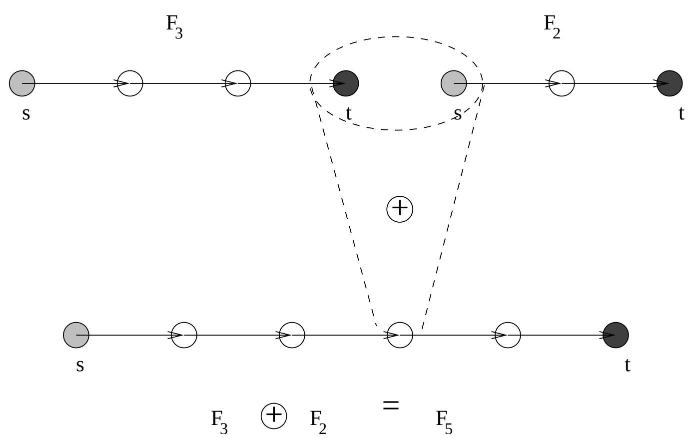

FIG. 1.1. *Interpreting addition of natural numbers inside $\mathcal{F}$.*

To extend this definition of $\bigoplus$ to all of F, we define general addition of flow graphs as follows: Given two flow graphs A and B, define $A \bigoplus B$ to be the flow graph obtained by identifying t_A with s_B and defining $s_{A \bigoplus B} = s_A$ and $t_{A \bigoplus B} = t_B$. An example of such an addition is shown in Figure 1.2.

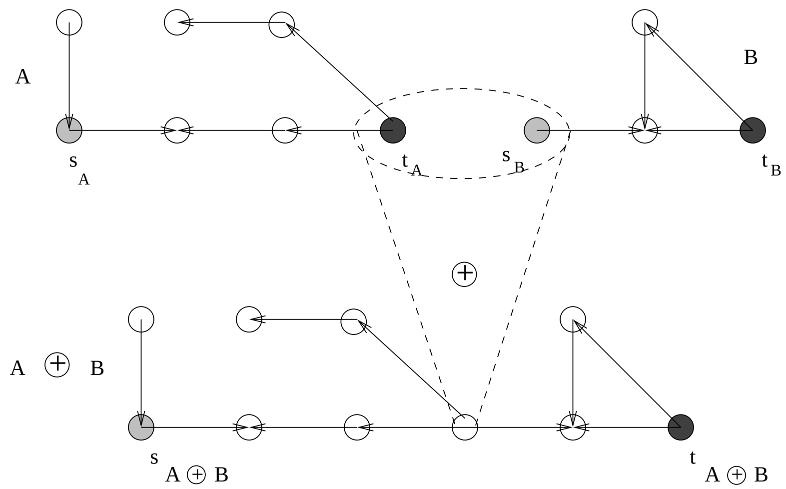

FIG. 1.2. *General addition of flow graphs.*

To make this formal we define the following operation on connected directed multigraphs: Given directed graphs G_1 and G_2, and vertices $u_1 \in V[G_1]$, $u_2 \in V[G_2]$, we define

$$G_1 \bigoplus {}_{u_1 \approx u_2} G_2 \stackrel{\text{def}}{=} (G_1 \sqcup G_2)/(u_1 \approx u_2)$$

to be the graph obtained by taking disjoint copies of G_1 and G_2 and identifying vertex u_1 in G_1 with vertex u_2 in G_2. Note the obvious and natural injective graph homomorphisms

$$\begin{array}{rcl} \sigma^{\bigoplus}_{u_1 \approx u_2} : G_1 & \hookrightarrow & G_1 \bigoplus {}_{u_1 \approx u_2} G_2 \\ \tau^{\bigoplus}_{u_1 \approx u_2} : G_2 & \hookrightarrow & G_1 \bigoplus {}_{u_1 \approx u_2} G_2 . \end{array} \tag{1.1}$$

DEFINITION 1.3. *Given two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$, we define*

$$A \bigoplus B \stackrel{\text{def}}{=} (G_A \bigoplus {}_{t_A \approx s_B} G_B, s_A, t_B).$$

REMARK 1.4. *Note that if A is a flow graph with p_A vertices and q_A edges, and B is a flow graph with p_B vertices and q_B edges, then the number of vertices and edges in $A \bigoplus B$ is $p_A + p_B - 1$ and $q_A + q_B$, respectively.*

The proofs of the following two lemmas follow immediately from the definition of $\bigoplus$ and Remark 1.4.

LEMMA 1.5. *Let m, n be natural numbers. Then $i(n+m) = i(n) \bigoplus i(m)$.*

LEMMA 1.6. *$⓪ \stackrel{\text{def}}{=} F_0$ is the unique two-sided identity with respect to $\bigoplus$. That is, for all flow graphs $A, G \in \mathcal{F}$,*

$$A \bigoplus G = A \quad \Leftrightarrow \quad G = ⓪ \quad \Leftrightarrow \quad G \bigoplus A = A.$$

DEFINITION 1.7 (Scalar multiplication of flow graphs). *Given a flow graph A, and a positive natural number k in $\mathbb{N}$, we define left-multiplication inductively as follows:*

$$1A = A$$
$$kA = (k-1)A \bigoplus A.$$

Right-multiplication is defined analogously. However, as we will see, $\bigoplus$ is associative, and so the two notions coincide. We shall subsequently consider only left-multiplication by integer scalars.

1.2. Multiplication. In the previous section, we presented an interpretation of addition in $\mathcal{F}$ that is a natural extension of addition on the natural numbers. In this section, we give an interpretation of multiplication in $\mathcal{F}$. In doing this, we must respect the fact that for each pair of natural numbers n_1, n_2, the following identity holds in $\mathcal{N}$:

$$I_{n_1,n_2} : \underbrace{n_2 + n_2 + \cdots + n_2}_{n_1 \text{ times}} = n_1 n_2 = \underbrace{n_1 + n_1 + \cdots + n_1}_{n_2 \text{ times}}.$$

So, in particular, the definition of $\bigotimes$ in $\mathcal{F}$ must satisfy

$$n_1 F_{n_2} = F_{n_1} \bigotimes F_{n_2} = n_2 F_{n_1}. \tag{1.2}$$

Given that we represent the natural number n by the flow graph F_n, the product of two graphical numbers F_{n_1} and F_{n_2} can be made to satisfy relation (1.2) if we take multiplication to be the act of replacing each edge of F_{n_1} with a copy of F_{n_2}. Consider the multiplication of graphical natural numbers F_3 and F_2, as depicted in Figure 1.3.

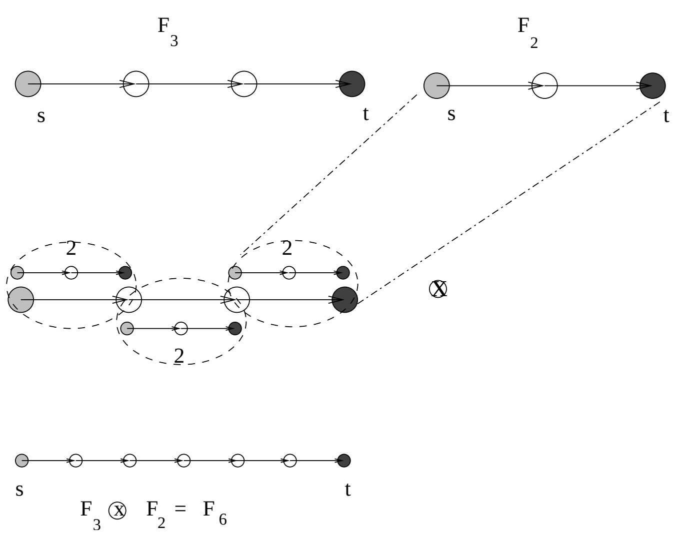

FIG. 1.3. *Standard multiplication of natural numbers in $\mathcal{F}$ (represented as flow graphs).*

To extend this definition of $\bigotimes$ to all of F, we define general multiplication of flow graphs as follows: Given two flow graphs A and B, define $A \bigotimes B$ to be the flow graph obtained by replacing every edge e (from $E[G_A]$) with a copy of B as follows:

For each edge $e = (u, v)$ in A, we remove e and replace it with a graph B_e isomorphic to B, by identifying u with s_{B_e}, and v with t_{B_e}. An example of such a multiplication is shown in Figure 1.4.

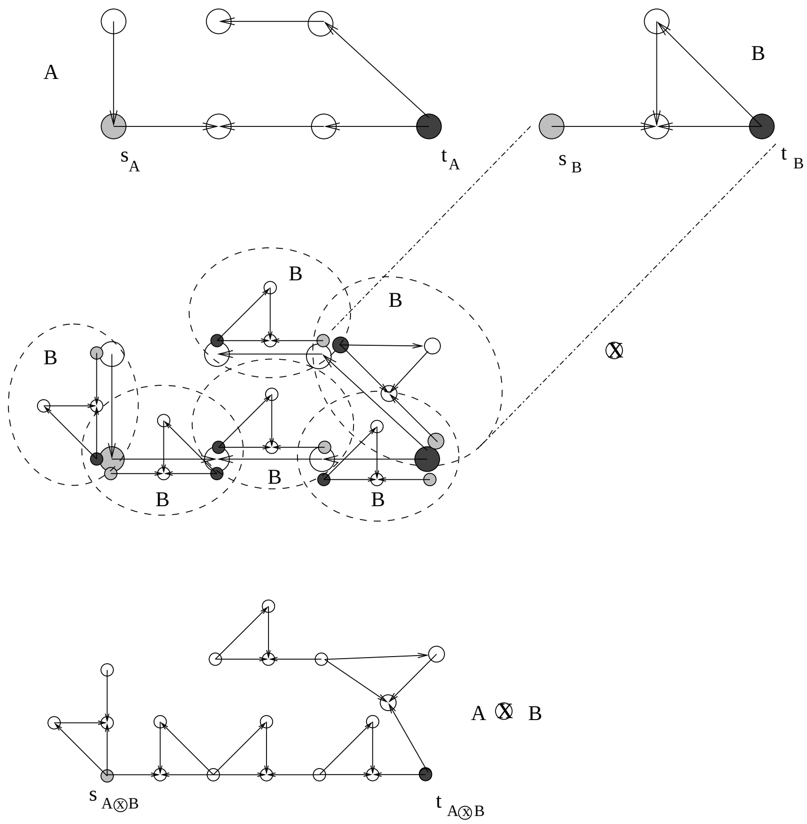

FIG. 1.4. *General multiplication of flow graphs.*

To make this formal we define the following operation on connected directed multigraphs: Given directed graphs G_1 and G_2, an edge $e = (u_1, v_1) \in E[G_1]$ and vertices $u_2, v_2 \in V[G_2]$, we define

$$G_1 \otimes_{e\approx(u_2,v_2)} G_2 \stackrel{\text{def}}{=} [(G_1 \backslash e) \sqcup G_2]/(u_1 \approx u_2, v_1 \approx v_2)$$

to be the graph obtained by removing e from G_1 and attaching a copy of G_2 to the resulting graph by gluing u_1 with u_2 and v_1 with v_2. Note the obvious and natural injective maps

$$\begin{aligned} \sigma^{\otimes}_{e\approx(u_2,v_2)} : G_1 \backslash e &\hookrightarrow G_1 \otimes_{e\approx(u_2,v_2)} G_2 \\ \tau^{\otimes}_{e\approx(u_2,v_2)} : G_2 &\hookrightarrow G_1 \otimes_{e\approx(u_2,v_2)} G_2. \end{aligned} \tag{1.3}$$

DEFINITION 1.8. *Given flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$. Define the directed graph $K_0 = G_A$, $E_0 = E[G_A]$, and let $\bar{\sigma}_0, \sigma_0 : K_0 \to K_0$ be the identity isomorphisms. Fix any enumeration η of the edges $E[G_A]$, say $\eta =$*

$e_1, e_2, \ldots, e_m$. *Inductively, for* $i = 1, 2, \ldots, m$ *we define*

$$\begin{aligned} K_i &= K_{i-1} \otimes_{\bar{\sigma}_{i-1}(e_i) \approx (s_B, t_B)} G_B \\ \sigma_i &= \sigma^{\otimes}_{\bar{\sigma}_{i-1}(e_i) \approx (s_B, t_B)} : K_{i-1} \backslash e_i \to K_i \\ E_i &= E_{i-1} \backslash \{e_i\} \\ \bar{\sigma}_i &= (\sigma_i)_{|E_i} (\sigma_{i-1})_{|E_i} \cdots (\sigma_1)_{|E_i} (\sigma_0)_{|E_i}. \end{aligned}$$

Informally, K_i *is the directed graph obtained after edges* $e_1, \ldots, e_i$ *have been deleted from* G_A *and replaced by copies of* G_B. *Finally, we put*

$$A \otimes_\eta B \stackrel{def}{=} (K_m, \bar{\sigma}_m(s_A), \bar{\sigma}_m(t_A)).$$

The reader may verify that the operation $\otimes_\eta$ is well-defined, and that in particular, it is independent of the chosen enumeration η of the edges $E[G_A]$.

REMARK 1.9. *Note that if* A *has* p_A *nodes and* q_A *edges and (non-trivial)* B *has* p_B *nodes and* q_B *edges then it can be verified that the number of nodes and edges in* $A \otimes B$ *is equal to* $p_A + q_A(p_B - 2)$ *and* $q_A q_B$, *respectively.*

The proofs of the following two lemmas follow immediately from the definition of $\otimes$ and Remark 1.9.

LEMMA 1.10. *Let* m, n *be natural numbers. Then* $i(n \times m) = i(n) \otimes i(m)$.

LEMMA 1.11. $① \stackrel{def}{=} F_1$ *and* $⓪$ *are the unique two-sided identity and two-sided annhilator for* $\otimes$, *respectively. That is, given flow graphs* G *and* H, *and a non-trivial flow graph* A*:*

$$\begin{array}{ccccc} A \otimes G = A & \Leftrightarrow & G = ① & \Leftrightarrow & G \otimes A = A, \\ G \otimes H = ⓪ & \Leftrightarrow & H = ⓪ & or & G = ⓪. \end{array}$$

DEFINITION 1.12 (Scalar exponentiation of flow graphs). *Given a flow graph* A, *and a positive natural number* k *in* $\mathbb{N}$, *we define right-exponentiation inductively as follows:*

$$\begin{aligned} A^1 &= A \\ A^k &= A^{k-1} \otimes A. \end{aligned}$$

Left-exponentiation is defined analogously. However, as we will see shortly, $\otimes$ *is associative, and so the two notions coincide. We shall subsequently consider only right-exponentiation by integer scalars.*

1.3. Order. Given our representation of the natural number n by the flow graph F_n in Definition 1.2, comparing the order of two numbers n_1 and n_2 amounts to simply comparing the lengths of the corresponding chain graphs F_{n_1} and F_{n_2}. To generalize this to all of $\mathcal{F}$, however, we cannot refer to "length". In what follows, we present two possible interpretations of $\leqslant$ in $\mathcal{F}$. To avoid confusion, we refer to these interpretations as $\circledleq$ and $\circledleq$.

1.3.1. Weak Order $\circledleq$**.** Suppose we are given two flow graphs A and B. Informally, we say that $A \circledleq B$ iff there is a way to partition A into edge-disjoint neighborhoods of the source/target of vertices of A in such a way that these neighborhoods can be mapped into disjoint neighborhoods of the source/target vertices of B. To make this more precise we define the following operation on connected directed multigraphs.

DEFINITION 1.13 ((s,t)-splitting). *Given a connected directed multigraph* $G = (V, E)$ *and two vertices* s *and* t *in* V, *an* (s,t)-splitting *of* G *is a pair of graphs* (H_1, H_2) *with the following properties:*

- H_1 *and* H_2 *are connected subgraphs of* G.
- s *is in* $V[H_1]$ *and* t *is in* $V[H_2]$.
- $\{E[H_1], E[H_2]\}$ *is a partition of* E. *While this implies* $V[H_1] \cup V[H_2] = V[G]$, *we remark that* $V[H_1] \cap V[H_2]$ *need not be empty.*

We can now give a precise definition of the weak ordering.

DEFINITION 1.14 (Weak order). *Given two flow graphs* $A = (G_A, s_A, t_A)$ *and* $B = (G_B, s_B, t_B)$, *we say that* $A \textcircled{\leqslant} B$ *if there is an* (s_A, t_A)*-splitting* (H_1, H_2) *of* G_A *and graph embeddings* $\phi_1 : H_1 \to G_B$, $\phi_2 : H_2 \to G_B$ *such that* $\phi_1(s_A) = s_B$ *and* $\phi_2(t_A) = t_B$ *and* $\phi_1(E[H_1]) \cap \phi_2(E[H_2]) = \emptyset$.

Consider the comparison of F_3 and F_5 in Figure 1.5 which illustrates the assertion that $F_3 \textcircled{\leqslant} F_5$.

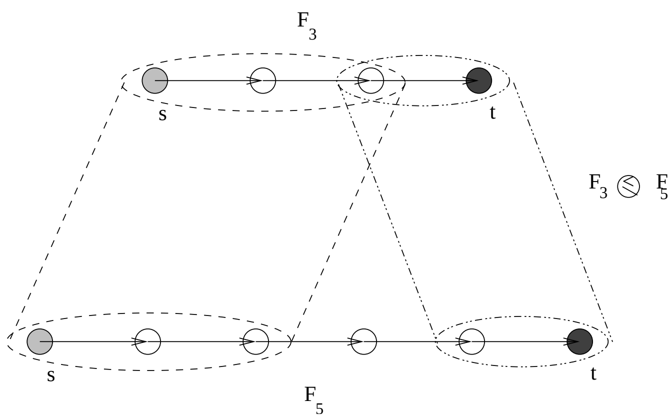

FIG. 1.5. *Standard weak ordering of natural numbers (represented as flow graphs).*

The proof of the following lemma is immediate.

LEMMA 1.15. *Let* m, n *be natural numbers. Then* $n \leqslant m \Leftrightarrow i(n) \textcircled{\leqslant} i(m)$.

Figure 1.6 illustrates a more general example in which weak order is used to compare two elements of $\mathcal{F}$ which are *not* graphical natural numbers.

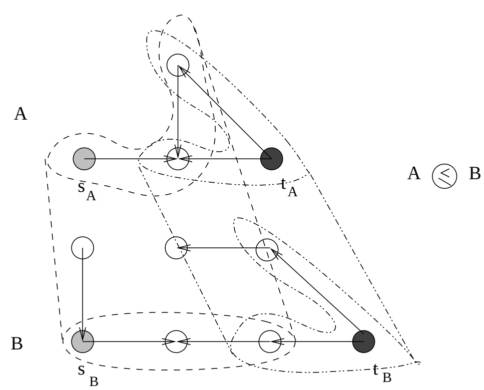

FIG. 1.6. *General weak ordering of flow graphs.*

The next Proposition follows immediately from Lemmas 1.5, 1.6, 1.10, 1.11, and 1.15.

PROPOSITION 1.16. *Under the embedding* $i : n \mapsto F_n$*, the standard model* $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant, +, \times \rangle$ *is a submodel of* $\mathcal{F} = \langle F, \textcircled{0}, \textcircled{1}, \textcircled{\leqslant}, \oplus, \otimes \rangle$*, where* $\textcircled{0} = F_0$*,* $\textcircled{1} = F_1$*, and the relations* $\oplus$*,* $\otimes$ *and* $\textcircled{\leqslant}$ *reinterpret* $+, \times$ *and* $\leqslant$ *inside* $\mathcal{F}$*.*

1.3.2. Strong Order $\textcircled{\leqslant}$**.** We now give an alternate, strengthened ordering on $\mathcal{F}$. Given two flow graphs A and B, informally, we say that $A \textcircled{\leqslant} B$ iff a copy of G_A appears as a neighborhood of both s_B and t_B in G_B. The next definition makes this statement precise.

DEFINITION 1.17 (Strong order). *Given two flow graphs* $A = (G_A, s_A, t_A)$ *and* $B = (G_B, s_B, t_B)$*, we say* $A \textcircled{\leqslant} B$ *iff there are graph embeddings* $\phi_s : G_A \to G_B$ *and* $\phi_t : G_A \to G_B$ *which satisfy* $\phi_s(s_A) = s_B$ *and* $\phi_t(t_A) = t_B$*.*

Consider the comparison of F_3 and F_5 depicted in Figure 1.7; clearly $F_3 \textcircled{\leqslant} F_5$.

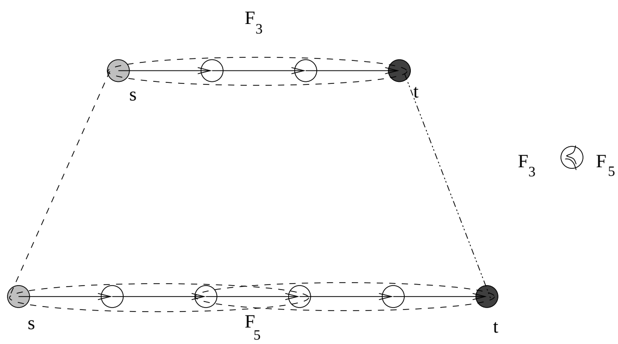

FIG. 1.7. *Standard strong ordering of natural numbers (represented as flow graphs).*

The proof of the following lemma is immediate.

LEMMA 1.18. *Let* m, n *be natural numbers. Then* $n \leqslant m \Leftrightarrow i(n) \textcircled{\leqslant} i(m)$*.*

Figure 1.8 illustrates a more general example in which strong order is used to compare two elements of $\mathcal{F}$ which are *not* graphical natural numbers.

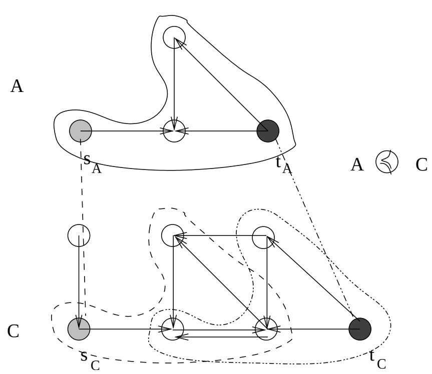

FIG. 1.8. *General strong ordering of flow graphs.*

The next Proposition follows immediately from Lemmas 1.5, 1.6, 1.10, 1.11, and 1.18.

PROPOSITION 1.19. *Under the embedding $i : n \mapsto F_n$, the standard model $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant, +, \times \rangle$ is a submodel of $\mathcal{F} = \langle F, \textcircled{0}, \textcircled{1}, \textcircled{\preccurlyeq}, \oplus, \otimes \rangle$, where $\textcircled{0} = F_0$, $\textcircled{1} = F_1$, and the relations $\oplus$, $\otimes$ and $\textcircled{\preccurlyeq}$ reinterpret $+, \times$ and $\leqslant$ inside $\mathcal{F}$.*

The next proposition and example show that ordering by $\textcircled{\preccurlyeq}$ is indeed strictly stronger than ordering by $\textcircled{\leqslant}$.

PROPOSITION 1.20. *Given flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$*

$$A \textcircled{\preccurlyeq} B \Rightarrow A \textcircled{\leqslant} B.$$

Proof. Since $A \textcircled{\preccurlyeq} B$, there are graph embeddings $\phi_s : G_A \to G_B$ and $\phi_t : G_A \to G_B$ which satisfy $\phi_s(s_A) = s_B$ and $\phi_t(t_A) = t_B$. Let $E_1 = E[G_A]$, $V_1 = V[G_A]$; take $E_2 = \emptyset$, $V_2 = \{t_A\}$. Put $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$. Then (H_1, H_2) is an (s_A, t_A)-splitting of G_A. We take graph embeddings $\phi_1 = \phi_s : H_1 \to G_B$, and $\phi_2 = \phi_t : H_2 \to G_B$. Then $\phi_1(s_A) = s_B$ and $\phi_2(t_A) = t_B$ and $\phi_1(E[H_1]) \cap \phi_2(E[H_2]) = \emptyset$. Thus, $A \textcircled{\leqslant} B$. □

The converse of Proposition 1.20 is false, as the following example indicates.

EXAMPLE 1.21. *Take A and B to be the flow graphs depicted on page 7, where Figure 1.6 illustrates that $A \textcircled{\leqslant} B$. Note that G_A contains a vertex of degree 3, while G_B does not, hence no neighborhood of s_B or t_B can be isomorphic to G_A. Thus $A \not\textcircled{\preccurlyeq} B$.*

2. Results. We begin by considering properties of $\oplus$ in Section 2.1. We show that $\oplus$ is an associative, non-commutative operation, and provide a natural criterion for a flow graph to be irreducible as a proper sum. We prove that every flow graph is canonically decomposable as a sum of irreducibles. Using this canonical decomposition, we deduce left and right cancellation laws for $\oplus$, and show that if two flow graphs A and B commute with respect to $\oplus$ then they are necessarily scalar multiples of some flow graph C. Then, in Section 2.2 we show that $\otimes$ is an associative, non-commutative operation and that it right-distributes over $\oplus$ (but does not left-distribute). We define left and right divisibility of flow graphs, and use this to introduce the notion of a prime flow graph, and show that the concept of left-prime and right-prime coincide. We describe the canonical $\oplus$ decomposition of flow graph products in terms of the $\oplus$ decompositions of each of the $\otimes$ factors. Finally, in Section 2.3, we explore the relationship between strong order (denoted by $\textcircled{\preccurlyeq}$) and weak order (denoted by $\textcircled{\leqslant}$), describing the interaction between these orders and the operations of $\oplus$ and $\otimes$. We show that while the two orders coincide on the graphical natural numbers, neither order is anti-symmetric on all of $\mathcal{F}$, and only $\textcircled{\preccurlyeq}$ is transitive. On the other hand, many of the laws that govern the relationship between $\leqslant$, $+$ and $\times$ in $\mathcal{N}$ continue to hold for $\textcircled{\leqslant}$, $\oplus$ and $\otimes$ in $\mathcal{F}$, but these laws are violated under the ordering $\textcircled{\preccurlyeq}$.

2.1. Additive Properties. In this section we present some properties of $\oplus$.

LEMMA 2.1 (Associativity of $\oplus$). *The operation $\oplus$ is associative.*

Proof. Given flow graphs A, B, C,

$$\begin{aligned}
(A \oplus B) \oplus C &= (G_A \oplus_{t_A \approx s_B} G_B, s_A, t_B) \oplus C \\
&= ((G_A \oplus_{t_A \approx s_B} G_B) \oplus_{t_B \approx s_C} G_C, s_A, t_C) \\
&= (G_A \oplus_{t_A \approx s_B} (G_B \oplus_{t_B \approx s_C} G_C), s_A, t_C) \\
&= A \oplus (G_B \oplus_{t_B \approx s_C} G_C, s_B, t_C) \\
&= A \oplus (B \oplus C).
\end{aligned}$$

□

EXAMPLE 2.2. *Let A be the flow graph consisting of a directed cycle of length* 3 *and let source and target vertices be any two vertices on this cycle. Then it is easy to check that $A \bigoplus F_2$ is not equal to $F_2 \bigoplus A$, that is to say, there is no flow graph isomorphism between $A \bigoplus F_2$ and $F_2 \bigoplus A$ (see Figure 2.1).*

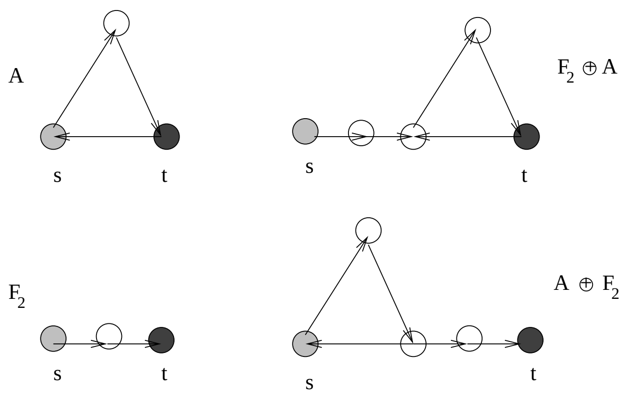

FIG. 2.1. *Example showing the non-commutativity of addition in $\mathcal{F}$.*

The previous example proves the next lemma.

LEMMA 2.3. *The operation $\bigoplus$ is not commutative.*

The next definition is the flow graph analogue of a cut vertex in standard graphs.

DEFINITION 2.4 (Splitting vertex for a flow graph). *We say that w is a* splitting vertex *for flow graph $A = (G_A, s_A, t_A)$ if $w \neq s_A, t_A$ and the deletion of w from G_A produces precisely* two *components, one containing s_A and the other containing t_A. We denote the component containing s_A as $G^s_A(w)$, and the one containing t_A as $G^t_A(w)$. Let $i_s : G^s_A(w) \hookrightarrow G_A$, $i_t : G^t_A(w) \hookrightarrow G_A$ denote the natural subgraph injections. Since w is a splitting vertex for A, $s_A \in Im(i_s)$, $t_A \in Im(i_t)$. The pair of graphs $(G^s_A(w), G^t_A(w))$ is called the (s_A, t_A) splitting of G_A* induced *by w.*

DEFINITION 2.5 (Flow graph splitting). *Suppose vertex w is a splitting vertex for flow graph $A = (G_A, s_A, t_A)$. Take $t_{A^w_s}$ to be a new vertex (not present in $V[G^s_A(w)]$), and define flow graph $A^w_s = (G_{A^w_s}, s_{A^w_s}, t_{A^w_s})$ as follows:*

$$\begin{aligned}
V[G_{A^w_s}] &= V[G^s_A(w)] \cup t_{A^w_s}, \\
E[G_{A^w_s}] &= E[G^s_A(w)] \\
&\quad \cup \{(t_{A^w_s}, u) \mid (w, u) \in E[G_A], u \in V[G^s_A(w)]\} \\
&\quad \cup \{(u, t_{A^w_s}) \mid (u, w) \in E[G_A], u \in V[G^s_A(w)]\}, \\
s_{A^w_s} &= i_s^{-1}(s_A).
\end{aligned}$$

Analogously, let $s_{A^w_t}$ be a vertex not present in $V[G^t_A(w)]$. Define A^w_t to be the flow graph $(G_{A^w_t}, s_{A^w_t}, t_{A^w_t})$ as follows.

$$\begin{aligned}
V[G_{A^w_t}] &= V[G^t_A(w)] \cup s_{A^w_t}, \\
E[G_{A^w_t}] &= E[G^t_A(w)] \\
&\quad \cup \{(s_{A^w_t}, u) \mid (w, u) \in E[G_A], u \in V[G^t_A(w)]\} \\
&\quad \cup \{(u, s_{A^w_t}) \mid (u, w) \in E[G_A], u \in V[G^t_A(w)]\}, \\
t_{A^w_t} &= i_t^{-1}(t_A).
\end{aligned}$$

The pair of flow graphs (A^w_s, A^w_t) *is referred to as the* splitting *of* A *induced by* w.

DEFINITION 2.6 ($\bigoplus$ -Irreducible). *A flow graph* A *is called* $\bigoplus$ -reducible *if can be expressed as a non-trivial sum* $A = B \bigoplus C$ *(where both* B *and* C *are non-trivial flow graphs). Otherwise it is called* $\bigoplus$ *-irreducible.*

LEMMA 2.7 ($\bigoplus$ -Irreducibility Lemma). *Flow graph* $A = (G_A, s_A, t_A)$ *is* $\bigoplus$ *-reducible if and only if* $V[G_A]$ *contains a splitting vertex for* A.

Proof. If $A = B \bigoplus C$, then

$$w = \sigma_{\bigoplus_{t_B \approx s_C}^{-1}}(t_B) = \tau_{\bigoplus_{t_B \approx s_C}^{-1}}(s_C)$$

is a splitting vertex for A (see expression (1.1) on page 3 for definitions of the σ and τ injections). Conversely, if w is a splitting vertex for A, then $A = A^w_s \bigoplus A^w_t$. □

PROPOSITION 2.8 (Component-wise decomposition of isomorphisms under $\bigoplus$). *Suppose* A *and* B *are two flow graphs, expressed as sums of* $\bigoplus$ *-irreducible flow graphs as follows:*

$$A = A_0 \bigoplus A_1 \bigoplus \cdots \bigoplus A_{m-1},$$
$$B = B_0 \bigoplus B_1 \bigoplus \cdots \bigoplus B_{n-1}$$

Then

$$A = B \Leftrightarrow m = n \text{ and } A_i = B_i \text{ for all } i = 0, \ldots, m-1.$$

Proof. [⇐] Let $\phi_i : A_i \to B_i$ be given component-wise isomorphisms, for $i = 0, \ldots, m-1$. Define $\phi : A \to B$ by defining $\phi|_{A_i} = \phi_i$. Since $t_{B_i} = \phi_i(t_{A_i}) = \phi_{i+1}(s_{A_{i+1}}) = s_{B_{i+1}}$ for $i = 0, \ldots, m-1$, this provides a well-defined isomorphism between A and B.

[⇒] We prove the statement by induction on $\max(m, n)$. In the case when $m = n = 1$, the claim is trivial. For the inductive step, let $\phi : A \to B$ be an isomorphism. Consider $\phi(A_0)$, and take k to be the smallest integer in $\{0, \ldots, n-1\}$ for which $\phi(A_0)$ is a subgraph of $B_0 \bigoplus B_1 \bigoplus \cdots \bigoplus B_k$. Since A_0 contains no splitting vertices, it must be that $k = 0$, since otherwise the $\bigoplus$ -irreducible flow graph A_0 would be isomorphic to the reducible flow graph $\phi(A_0)$. Since $k = 0$, we have shown that $\phi(A_0)$ is a subgraph of B_0. Now, repeating the argument for B_0 using ϕ^{-1}, we see that $\phi^{-1}(B_0)$ is a subgraph of A_0. It follows that A_0 is isomorphic to B_0 under a suitable restriction of ϕ. Now, since $\phi(A) = B$ and $\phi(A_0) = B_0$ it follows that $\phi(A \backslash A_0) = B \backslash B_0$, or more specifically $\phi(A_1 \bigoplus \cdots \bigoplus A_{m-1}) = B_1 \bigoplus \cdots \bigoplus B_{n-1}$. By inductive hypothesis, this implies that $m = n$ and $A_i = B_i$ for all $i = 1, \ldots, m-1$. □

DEFINITION 2.9 (Splitting vertex ranking). *Given flow graph* $A = (G_A, s_A, t_A)$, *let* $\chi(A) \subset V[G_A]$ *be the set of all splitting vertices for* A. *We define the* s-ranking *and* t-ranking *functions* $r^A_s, r^A_t : \chi(A) \to \mathbb{N}$ *as follows:*

$$r^A_s(w) = |V[G^s_A(w)] \cap \chi(A)|,$$
$$r^A_t(w) = |V[G^t_A(w)] \cap \chi(A)|.$$

When it is clear from the context, we denote $r_s(w) = r^A_s(w)$ *and* $r_t(w) = r^A_t(w)$.

LEMMA 2.10. *Let* $w \in \chi(A)$ *be a splitting vertex for flow graph* $A = (G_A, s_A, t_A)$. *Then for all* $u \in V[G^s_A(w)] \cap \chi(A)$*:*

$$r_s(u) < r_s(w),$$
$$r_t(u) > r_t(w);$$

and for all $u \in V[G^t_A(w)] \cap \chi(A)$*:*

$$r_s(u) > r_s(w),$$
$$r_t(u) < r_t(w).$$

Proof. First, note that for any u in $(V[G^s_A(w)] \cup V[G^t_A(w)]) \cap \chi(A)$

$$r_s(u) + r_t(u) + 1 = |\chi(A)| \tag{2.1}$$

Now if $u \in V[G^s_A(w)] \cap \chi(A)$, then since $w \in (V[G^t_A(u)] \backslash V[G^t_A(w)]) \cap \chi(A)$, so it follows that $V[G^t_A(w)] \subsetneq V[G^t_A(u)]$. But then $r_t(w) < r_t(u)$. By expression 2.1 above, it follows that $r_s(w) > r_s(u)$. The proof for the case when $u \in V[G^t_A(w)] \cap \chi(A)$ is analogous. □

LEMMA 2.11. *Given a flow graph* $A = (G_A, s_A, t_A)$*, for each* $i = 0, 1, \ldots, |\chi(A)| - 1$ *there is a unique vertex* v_i *in* $\chi(A)$ *with the property that* $r_s(v_i) = i$*.*

Proof. First we note that one cannot have two distinct vertices v, v' having $r_s(v) = r_s(v')$, since either $v \in V[G^s_A(v')]$ or $v' \in V[G^s_A(v)]$, and so by Lemma 2.10 it follows that $r_s(v) \neq r_s(v')$. Base case: $i = 0$. Let w_0 be any vertex in $\chi(A)$. If $r_s(w) > 0$, then $V[G^s_A(w)] \cap \chi(A)$ is not empty. So let w_1 be any vertex in $V[G^s_A(w)] \cap \chi(A)$. By Lemma 2.10, $r_s(w_1) < r_s(w_0)$. Repeating in this fashion, after finitely many steps $w_0 \rightsquigarrow w_1 \rightsquigarrow \ldots$ we find some vertex v_0 for which $r_s(v_0) = 0$. Inductive step $i+1$: Let v_i be the unique vertex in $\chi(A)$ having $r_s(v_i) = i$. Define v_{i+1} to be the vertex in $V[G^t_A(v_i)] \cap \chi(A)$ for whose s-rank is minimal. Since

$$V[G^s_A(v_{i+1})] \cap \chi(A) = [V[G^s_A(v_i)] \cap \chi(A)] \cup \{v_i\},$$

it follows that $r_s(v_{i+1}) = r_s(v_i) + 1 = i + 1$, hence the result. □

DEFINITION 2.12 (Canonical $\bigoplus$-decomposition). *Let* $A = (G_A, s_A, t_A)$ *be a flow graph. Take* $\chi(A) = \{v_0, v_1, \ldots, v_{|\chi(A)|-1}\}$ *to be the set of splitting vertices for* A*, ordered according to the indexing scheme postulated in Lemma 2.11. Define* $A^{(0)} = A^{v_0}_s$*,* $\bar{A}^{(0)} = A^{v_0}_t$*, and then for each* $i = 1, 2, \ldots, |\chi(A)| - 1$*, put*

$$A^{(i)} = (\bar{A}^{(i-1)})^{v_i}_s,$$
$$\bar{A}^{(i)} = (\bar{A}^{(i-1)})^{v_i}_t.$$

We shall denote $\bar{A}^{(|\chi(A)|-1)}$ *as* $A^{(|\chi(A)|)}$*. The* canonical $\bigoplus$-decomposition of A *is defined to be the sequence*

$$\langle A \rangle \stackrel{\text{def}}{=} (A^{(0)}, A^{(1)}, \ldots, A^{(|\chi(A)|-1)}, A^{(|\chi(A)|)}).$$

Note that the effectiveness of this definition guarantees uniqueness of the decomposition.

LEMMA 2.13 ($\bigoplus$-decompositions for sums). *Given* $A = (G_A, s_A, t_A)$ *and* $B = (G_B, s_B, t_B)$*, two flow graphs with their respective canonical* $\bigoplus$*-decompositions* $\langle A \rangle$ *and* $\langle B \rangle$*. Then the canonical* $\bigoplus$*-decomposition of* $A \bigoplus B$ *is* $\langle A \rangle \langle B \rangle$*, the concatenation of* $\langle A \rangle$ *with* $\langle B \rangle$*.*

Proof. First, note that $|\chi(A \bigoplus B)| = |\chi(A)| + |\chi(B)| + 1$. More specifically, if

$$\chi(A) = \{u_0, u_1, \ldots, u_{|\chi(A)|-1}\}, \text{ and}$$
$$\chi(B) = \{v_0, v_1, \ldots, v_{|\chi(B)|-1}\}$$

are the sets of splitting vertices for A and B respectively, ordered by ascending s-rank, according to the indexing scheme postulated in Lemma 2.11, then $A \oplus B$ has splitting vertices:

$$\chi(A \oplus B) = \Big\{\sigma^{\oplus}_{t_A \approx s_B}(u_0), \sigma^{\oplus}_{t_A \approx s_B}(u_1), \ldots, \sigma^{\oplus}_{t_A \approx s_B}(u_{|\chi(A)|-1}),$$
$$\sigma^{\oplus}_{t_A \approx s_B}(t_A) = \tau^{\oplus}_{t_A \approx s_B}(s_B),$$
$$\tau^{\oplus}_{t_A \approx s_B}(v_0), \tau^{\oplus}_{t_A \approx s_B}(v_1), \ldots, \tau^{\oplus}_{t_A \approx s_B}(v_{|\chi(B)|-1})\Big\}.$$

But since $\sigma^{\oplus}_{t_A \approx s_B}$ and $\tau^{\oplus}_{t_A \approx s_B}$ are injections,

$$(A \oplus B)^{(i)} = \begin{cases} A^{(i)} & \text{for } 0 \leqslant i < |\chi(A)|, \\ \bar{A}^{(|\chi(A)|-1)} & \text{for } i = |\chi(A)|, \\ B^{(i-|\chi(A)|-1)} & \text{for } |\chi(A)| < i \leqslant |\chi(A)| + |\chi(B)|, \\ \bar{B}^{(|\chi(B)|-1)} & \text{for } i = |\chi(A)| + |\chi(B)| + 1. \end{cases}$$

It follows that $\langle A \oplus B\rangle = \langle A\rangle\langle B\rangle$. □

PROPOSITION 2.14 (Correctness of the $\oplus$-decomposition). *Consider the canonical $\oplus$-decomposition of A as given in Definition 2.12:*

$$\langle A^{(0)}, A^{(1)}, \ldots, A^{(|\chi(A)|)}\rangle.$$

Then

$$A = A^{(0)} \oplus A^{(1)} \oplus A^{(2)} \cdots \oplus A^{(|\chi(A)|-1)} \oplus A^{(|\chi(A)|)},$$

and every summand is $\oplus$-irreducible.

Proof. Since $r_s(v_i) = i$, it follows that $\chi(A^{(i)}) = \emptyset$ for all $i = 0, 1, \ldots, |\chi(A)|$. Since each summand has no splitting vertices, by Lemma 2.7, each is $\oplus$-irreducible. We prove the Proposition by induction on $|\chi(A)|$. The base case when $|\chi(A)| = 1$ is straightforward, since Definition 2.12 specified $A^{(0)} = A^{v_0}_s$ and $\bar{A}^{(0)} = A^{v_0}_t$. Then, since $A^{v_0}_s \oplus A^{v_0}_t = A$ for any splitting vertex v_0, the result follows. Suppose the Proposition has been proved for all flow graphs B which enjoy $|\chi(B)| \leqslant k$. Let A be a flow graph with $|\chi(A)| = k+1$. Unravelling Definition 2.12 yields $A^{(k+1)} = \bar{A}^{(k)} = (\overline{A}^{(k-1)})^{v_k}_t = A^{v_k}_t$. Then since $A = A^{v_k}_s \oplus A^{v_k}_t$ and $|\chi(A^{v_k}_s)| = k$, by inductive hypothesis and Lemma 2.13,

$$\begin{aligned} A &= A^{v_k}_s \oplus A^{v_k}_t \\ &= \langle A^{v_k}_s\rangle \oplus A^{v_k}_t \\ &= (A^{v_k}_s)^{(0)} \oplus \ldots (A^{v_k}_s)^{(k)} \oplus A^{v_k}_t \\ &= A^{(0)} \oplus A^{(1)} \oplus A^{(2)} \cdots \oplus A^{(k)} \oplus A^{(k+1)} \\ &= \langle A\rangle. \end{aligned}$$

The result follows. □

LEMMA 2.15 (Right-cancellation law for $\oplus$). *Let A, B, C be flow graphs.*

$$A \oplus B = A \oplus C \Rightarrow B = C.$$

Proof. Let $\langle A \bigoplus B\rangle = \langle A\rangle\langle B\rangle$ be the canonical $\bigoplus$-decomposition of $A \bigoplus B$, and $\langle A \bigoplus C\rangle = \langle A\rangle\langle C\rangle$ be the canonical $\bigoplus$-decomposition of $A \bigoplus C$ respectively. Proposition 2.14 indicates that any isomorphism $\phi : A \bigoplus B \to A \bigoplus C$ is also an isomorphism from the summation of the entries in $\langle A\rangle\langle B\rangle$ to the summation of the entries in $\langle A\rangle\langle C\rangle$. By Proposition 2.8, ϕ must send the summation of the initial entries $\langle A\rangle$ in the first decomposition to the summation of the initial entries $\langle A\rangle$ in the second decomposition. Thus ϕ must send the summation of the entries of $\langle B\rangle$ in the first decomposition to the summation of the entries of $\langle C\rangle$ in the second decomposition. Thus by Proposition 2.14, $B = C$. □

The next lemma is proved in a manner analogous to Lemma 2.15.

LEMMA 2.16 (Left-cancellation law for $\bigoplus$). *Let A, B, C be flow graphs.*

$$B \bigoplus A = C \bigoplus A \Rightarrow B = C.$$

PROPOSITION 2.17 (Commutativity condition for $\bigoplus$). *Given flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$,*

$$A \bigoplus B = B \bigoplus A$$

iff there exists a flow graph C and integers k_1, k_2 in $\mathbb{N}$ such that

$$A = k_1 C, \text{ and}$$
$$B = k_2 C.$$

Proof. [⇐] If $A = k_1C$ and $B = k_2C$, then $A \bigoplus B = (k_1 + k_2)C = B \bigoplus A$.

[⇒] The proof is carried by induction on $\max(|\chi(A)|, |\chi(B)|)$. Consider the canonical decompositions of A and B,

$$A = A^{(0)} \bigoplus A^{(1)} \bigoplus A^{(2)} \bigoplus \cdots \bigoplus A^{(|\chi(A)|-1)} \bigoplus A^{(|\chi(A)|)},$$
$$B = B^{(0)} \bigoplus B^{(1)} \bigoplus B^{(2)} \bigoplus \cdots \bigoplus B^{(|\chi(B)|-1)} \bigoplus B^{(|\chi(B)|)}.$$

If $|\chi(A)| = |\chi(B)|$ then Proposition 2.8 tells us that an isomorphism $\phi : A \bigoplus B \to B \bigoplus A$ restricts on the first summand A to yield an isomorphism from A to B. So in this case, we can take $C = A = B$ and $k_1 = k_2 = 1$. This proves the case $\max(|\chi(A)|, |\chi(B)|) = 0$, which forms the basis of the induction.

Suppose that $\max(|\chi(A)|, |\chi(B)|) > 0$, and $|\chi(A)| \neq |\chi(B)|$. Without loss of generality, suppose $|\chi(A)| < |\chi(B)|$. Then $A^{(i)} = B^{(i)}$ for $i = 0, \ldots |\chi(A)|$. It follows that

$$B^{i+(|\chi(A)|+1)} = B^{(i)} \text{ for } i = 0, \ldots, |\chi(B)| - (|\chi(A)| + 1), \tag{2.2}$$
$$B^{i-(|\chi(A)|+1)} = B^{(i)} \text{ for } i = (|\chi(A)| + 1), \ldots, |\chi(B)|. \tag{2.3}$$

If $(|\chi(B)| + 1)$ is divisible by $(|\chi(A)| + 1)$, then expressions (2.2) and (2.3) above are in fact equivalent, and in this setting, we take $C = A$, $k_1 = 1$ and $k_2 = \frac{(|\chi(B)|+1)}{(|\chi(A)|+1)}$ in order to satisfy the proposition. Suppose now that $(|\chi(B)| + 1)$ is *not* divisible by $(|\chi(A)| + 1)$. Put

$$d = \left\lfloor \frac{(|\chi(B)| + 1)}{(|\chi(A)| + 1)} \right\rfloor, \text{ and}$$
$$r = (|\chi(B)| + 1) \mod (|\chi(A)| + 1).$$

and define

$$X = B^{(0)} \bigoplus B^{(1)} \cdots \bigoplus B^{(r-1)},$$
$$Y = B^{(r)} \bigoplus B^{(r+1)} \cdots \bigoplus B^{(|\chi(A)|)}.$$

Note that $B = dA \bigoplus X$ and

$$\begin{aligned} A &= A^{(0)} \bigoplus A^{(1)} \bigoplus A^{(2)} \bigoplus \cdots \bigoplus A^{(|\chi(A)|-1)} \bigoplus A^{(|\chi(A)|)} \\ &= B^{(0)} \bigoplus B^{(1)} \bigoplus \cdots B^{(r-1)} \bigoplus B^{(r)} \bigoplus B^{(r+1)} \cdots \bigoplus B^{(|\chi(A)|-1)} \bigoplus B^{(|\chi(A)|)} \\ &= X \bigoplus Y. \end{aligned}$$

It follows that $B = d(X \bigoplus Y) \bigoplus X$. On the other hand, $X \bigoplus Y = A = Y \bigoplus X$ (see Figure 2.2), since

$$\begin{aligned} X \bigoplus Y &= A \\ &= A^{(0)} \bigoplus A^{(1)} \bigoplus \cdots \bigoplus A^{(|\chi(A)|)} \\ &= B^{(0)} \bigoplus B^{(1)} \bigoplus \cdots \bigoplus B^{(r)} \bigoplus \cdots \bigoplus B^{(|\chi(A)|-1)} \bigoplus B^{(|\chi(A)|)} \\ &= B^{(r)} \bigoplus \cdots \bigoplus B^{(|\chi(B)|-1)} \bigoplus B^{(|\chi(B)|)} \bigoplus B^{(0)} \bigoplus \cdots \bigoplus B^{(r-1)} \\ &= Y \bigoplus X. \end{aligned}$$

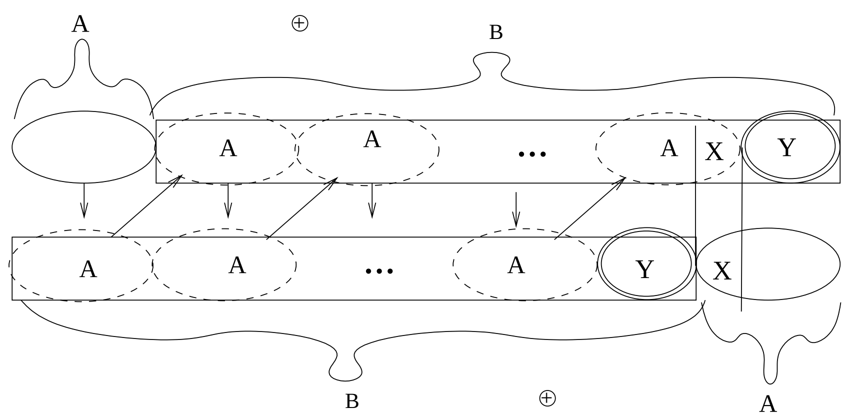

FIG. 2.2. *Inductive step showing* $X \bigoplus Y = A = Y \bigoplus X$.

Since $r \neq 0$ the inductive hypothesis applies to the flow graphs X, Y, i.e. there exists some flow graph Z and suitable integers l_1, l_2 so that $X = l_1 Z$, $Y = l_2 Z$. It follows that

$$A = X \bigoplus Y = l_1 Z \bigoplus l_2 Z = (l_1 + l_2)Z \text{ and}$$
$$B = dA \bigoplus X = d(X \bigoplus Y) \bigoplus X = d(l_1 + l_2)Z \bigoplus l_1 Z = ((d+1)l_1 + l_2)Z.$$

So taking $C = Z$, $k_1 = l_1 + l_2$ and $k_2 = (d+1)l_1 + l_2$, the Proposition is proved. ▯

2.2. Multiplicative Properties. In this section we present properties of $\bigotimes$.

LEMMA 2.18. *Given flow graphs A and B, there is a natural bijective correspondence*

$$\Lambda_{A,B} : E[A \bigotimes B] \to E[G_A] \times E[G_B]$$

Proof. Fix an edge e in $E[G_A \otimes G_B]$. Then e appears in $(A \otimes_\eta B)$ at some stage i where $1 \leqslant i \leqslant |E[G_A]|$ (where η is the enumeration specified in Definition 1.8). We define $\lambda_A(e)$ to be the edge $e_i \in E[G_A]$. At stage i we effectively replace edge $e_i = (u_i, v_i)$ with a new disjoint copy of G_B–by gluing s_B with u_i and t_B with v_i. Thus the edge e corresponds to some edge $\lambda_B(e)$ in this new disjoint copy of G_B. The desired bijection $e \mapsto \Lambda_{A,B}(e) = (\lambda_A(e), \lambda_B(e))$ is thus obtained. Note that the bijection $\Lambda_{A,B}$ is independent of the enumeration η of the edges of $E[G_A]$ which appears in the definition of $A \otimes B$. □

LEMMA 2.19 (Associativity of $\otimes$). *The operation $\otimes$ is associative.*

Proof. Given flow graphs $A = (G_A, s_A, t_A)$, $B = (G_B, s_B, t_B)$, $C = (G_C, s_C, t_C)$, we want to show:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C).$$

By Lemma 2.18, the map $\Lambda_{A \otimes B,C}$ is a bijective correspondence between the edges of $(A \otimes B) \otimes C$ and $(E[G_A] \times E[G_B]) \times E[G_C]$. Likewise, the edges of $A \otimes (B \otimes C)$ are in bijective correspondence with $E[G_A] \times (E[G_B] \times E[G_C])$, via $\Lambda_{A,B \otimes C}$. Obviously $(E[G_A] \times E[G_B]) \times E[G_C]$ is in bijective correspondence with $E[G_A] \times (E[G_B] \times E[G_C])$ by the map $\pi : ((e_1, e_2), e_3) \mapsto (e_1, (e_2, e_3))$. Then the composite map

$$\mu = \Lambda_{A,B \otimes C} \circ \pi \circ \Lambda_{A \otimes B,C}$$

is an isomorphism of flow graphs which carries $(A \otimes B) \otimes C$ to $A \otimes (B \otimes C)$. □

EXAMPLE 2.20. *Let A be the flow graph consisting of a directed cycle of length 3 and let source and target vertices be any two vertices on this cycle. Then it is easy to check that there is no flow graph isomorphism between $A \otimes F_2$ and $F_2 \otimes A$ (see Figure 2.3).*

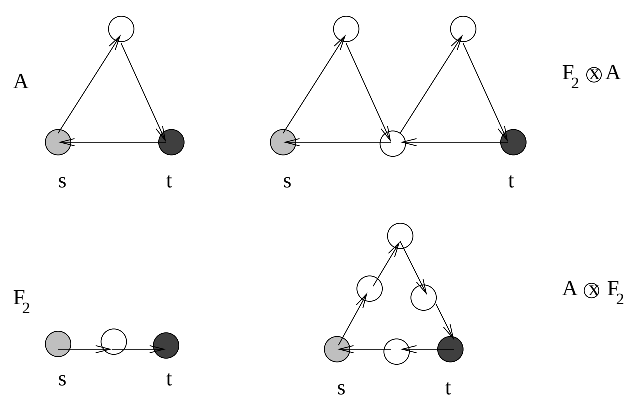

FIG. 2.3. *Example showing the non-commutativity of multiplication in $\mathcal{F}$.*

The previous example proves the next lemma.

LEMMA 2.21. *The operation $\otimes$ is not commutative.*

LEMMA 2.22 (Right-distributivity of $\otimes$ over $\oplus$). *For any flow graphs A, B, C,*

$$(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$$

Proof. Fix $e \in E[(A \oplus B) \otimes C]$. Then define $\beta_0(e) = \Lambda_{A \oplus B,C}(e)$. Note that $\beta_0(e) = (e', f)$, where e' is an edge in $E[G_{A \oplus B}]$ and f is one in $E[G_C]$. Define $\beta_1 : E[G_{A \oplus B}] \to E[G_A] \cup E[G_B]$ so that

$$\beta_1(e) = \begin{cases} \sigma^{\oplus\,-1}_{t_A \approx s_B}(e) & \text{if} \quad e \in Im(\sigma^{\oplus\,-1}_{t_A \approx s_B}) \\ \tau^{\oplus\,-1}_{t_A \approx s_B}(e) & \text{if} \quad e \in Im(\tau^{\oplus\,-1}_{t_A \approx s_B}). \end{cases}$$

Then $\beta_1 \circ \beta_0$ maps $E[(A \oplus B) \otimes C]$ injectively into $(E[G_A] \times E[G_C]) \cup (E[G_B] \times E[G_C])$. Define β_2 by taking

$$\beta_2(e) = \begin{cases} \Lambda^{-1}_{A,C}(e) & \text{if} \quad e \in E[G_{A \otimes C}] \\ \Lambda^{-1}_{B,C}(e) & \text{if} \quad e \in E[G_{B \otimes C}]. \end{cases}$$

Then β_2 maps $(E[G_A] \times E[G_C]) \cup (E[G_B] \times E[G_C])$ into $E[G_{A \otimes C}] \cup E[G_{B \otimes C}]$ injectively. Finally, define β_3 by taking

$$\beta_3(e) = \begin{cases} \sigma^{\oplus}_{t_{A \otimes C} \approx s_{B \otimes C}}(e) & \text{if} \quad e \in E[G_{A \otimes C}] \\ \tau^{\oplus}_{t_{A \otimes C} \approx s_{B \otimes C}}(e) & \text{if} \quad e \in E[G_{B \otimes C}]. \end{cases}$$

Then β_3 maps $E[G_{A \otimes C}] \cup E[G_{B \otimes C}]$ injectively into $E[G_{(A \otimes C) \oplus (B \otimes C)}]$. The composite map $\beta_3 \circ \beta_2 \circ \beta_1 \circ \beta_0$ maps the edges of $(A \oplus B) \otimes C$ injectively into the edges of $(A \otimes C) \oplus (B \otimes C)$, and is the desired flow graph isomorphism demonstrating the claimed equality. □

Let A be the flow graph consisting of a directed cycle of length 3 taking source and target vertices to be any two vertices on this cycle. Observe that $A \otimes (F_1 \oplus F_1) = A \otimes F_2$, while $(A \otimes F_1) \oplus (A \otimes F_1) = A \oplus A = 2A = F_2 \otimes A$. Referring to Figure 2.3 again, we see that $A \otimes F_2 \neq F_2 \otimes A$. Thus, we have shown

LEMMA 2.23 (Non Left-distributivity of $\otimes$ over $\oplus$). *There exist flow graphs A, B, C,*

$$A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$$

DEFINITION 2.24. *Given flow graphs A, B at least one of which is non-trivial, and a flow graph C, we say*

$$A/B = C \text{ iff } A = C \otimes B$$
$$A \backslash B = C \text{ iff } A = B \otimes C.$$

If there is no C for which $A/B = C$, we say that A/B does not exist and A is not right-divisible *by B. If there is no C for which $A \backslash B = C$, we say that $A \backslash B$ does not exist, and A is* not left-divisible *by B. By convention, we say that ⓪ / ⓪ and ⓪ \ ⓪ are undefined.*

Clearly if m and n are standard integers then F_m/F_n iff $F_m \backslash F_n$ iff m is divisible by n.

LEMMA 2.25. *For all flow graphs A, B, C*

$$A/C = (A/B) \otimes (B/C)$$
$$A \backslash C = (B \backslash C) \otimes (A \backslash B)$$

whenever these graphs exist.

Proof. Suppose $A/C = K_1$ and $B/C = K_2$. By definition, $A = K_1 \otimes B$ and $B = K_2 \otimes C$. Thus, $A = K_1 \otimes (K_2 \otimes C)$, which by Lemma 2.19 is $(K_1 \otimes K_2) \otimes C$. Thus A/C exists and equals $K_1 \otimes K_2 = (A/B) \otimes (B/C)$. Suppose $A\backslash C = K_1$ and $B\backslash C = K_2$. Then by definition, $A = B \otimes K_1$ and $B = C \otimes K_2$. Thus, $A = (C \otimes K_2) \otimes K_1$, which by Lemma 2.19 is $C \otimes (K_2 \otimes K_1)$. Thus $A\backslash C$ exists and equals $K_2 \otimes K_1 = (B\backslash C) \otimes (A\backslash B)$. □

LEMMA 2.26 (Distributivity of right-divisibility over $\oplus$). *For all flow graphs A, B, C,*

$$A/B \oplus C/B = (A \oplus C)/B$$

Proof. Suppose $A/B = K_1$ and $C/B = K_2$. Then by definition, $A = K_1 \otimes B$ and $C = K_2 \otimes B$. Thus $A \oplus C = (K_1 \otimes B) \oplus (K_2 \otimes B)$ which by Lemma 2.22, equals $(K_1 \oplus K_2) \otimes B$. It follows that $(A \oplus C)/B$ equals $K_1 \oplus K_2$, which is $A/B \oplus C/B$. □

OBSERVATION 2.27 (Non-distributivity of left-divisibility over $\oplus$). *Note that Lemma 2.23 can be used to construct examples that demonstrate non-distributivity of left-divisibility over $\oplus$. For example, let B be a directed cycle of length* 3 *with any two vertices as s_B and t_B. Take $A = B \otimes F_2$. Then $A\backslash B = F_2$. Now take $C = B$. Then $C\backslash B = F_1$ and so $(A\backslash B) \oplus (C\backslash B) = F_2 \oplus F_1 = F_3$. Since $A \oplus C \neq B \otimes F_3$, we see that $(A \oplus C)\backslash B \neq (A\backslash B) \oplus (C\backslash B)$.*

In Definition 2.4, we introduced the notion of a splitting vertex. Now, in unravelling information about $\otimes$, we require the notion of a splitting edge.

DEFINITION 2.28 (Splitting edge for a flow graph). *Let $A = (G_A, s_A, t_A)$ be a flow graph. A* splitting edge *of A is an edge $e \in E[G_A]$ with the property that $G\backslash e$ has precisely two components, one of which contains s_A and the other contains t_A. We denote the set of all splitting edges in A as $\Delta(A)$.*

LEMMA 2.29. *For any flow graph A, if A is $\oplus$-irreducible and $A \neq$ ①, then $\Delta(A) = \emptyset$.*

Proof. If $A \neq F_1$ and $e = (u, v)$ is a splitting edge then either u or v or both must be a splitting vertex. Hence A is $\oplus$-reducible. □

LEMMA 2.30 (Splitting edges in $\oplus$-decompositions). *Given a flow graph A, let $\langle A \rangle$ be its $\oplus$-decomposition. Then there exists a map $i : \Delta(A) \to \{0, 1, \dots, \chi(A)\}$ which injectively associates to every splitting edge an $\oplus$-irreducible component in the $\oplus$-decomposition of A, such that $A^{i(e)}$ is a component consisting only of edge e, and is isomorphic to F_1.*

Proof. Appealing to Proposition 2.14, fix ϕ an isomorphism from A to the component decomposition of $\langle A \rangle$. By Proposition 2.14, every $A^{(i_e)}$ is $\oplus$-irreducible. By Lemma 2.29 it either has no splitting edges or it is F_1. Suppose e is a splitting edge in A. Then $\phi(e)$ is a splitting edge inside $A^{(i_e)}$ for some i_e in $\{0, \dots, |\chi(A)|\}$. It follows that $A^{(i_e)} = F_1$ and the map $i : e \mapsto i_e$ enjoys the property claimed in the lemma. □

REMARK 2.31. *Given flow graphs A and B, a splitting vertex in $A \otimes B$ comes either from a splitting vertex of A or from a splitting vertex in (a copy of) B which lies on a splitting edge of A.*

OBSERVATION 2.32 ($\oplus$-decompositions for products). *Given flow graphs A, B,*

$$\begin{aligned}\langle A \otimes B \rangle &= (A^{(0)} \oplus A^{(1)} \oplus \cdots \oplus A^{(\chi(A))}) \otimes B \\ &= (A^{(0)} \otimes B) \oplus (A^{(1)} \otimes B) \oplus \cdots \oplus (A^{(\chi(A))} \otimes B) \\ &= \langle A^{(0)} \otimes B \rangle \langle A^{(1)} \otimes B \rangle \cdots \langle A^{(\chi(A))} \otimes B \rangle.\end{aligned}$$

Since $A^{(i)}$ *is* $\bigoplus$ *-irreducible, by Lemma 2.7 we know that* $\chi(A^{(i)}) = \emptyset$. *Case (i).* $A^{(i)} \neq F_1$. *Then by Lemma 2.14,* $\Delta(A^{(i)}) = \emptyset$. *It follows from Remark 2.31 that* $A^{(i)} \otimes B$ *has no splitting vertices, so by Lemma 2.7, it is* $\bigoplus$ *-irreducible. Hence,* $\langle A^{(i)} \otimes B\rangle$ *is a one-element sequence consisting of* $A^{(i)} \otimes B$. *Case (ii).* $A^{(i)} = F_1$. *Then* $A^{(i)} \otimes B = B$, *so* $A^{(i)} \otimes B$ *is* $\bigoplus$ *-irreducible iff* B *is* $\bigoplus$ *-irreducible. In this case* $\langle A^{(i)} \otimes B\rangle = \langle B\rangle$ *is a* $(|\chi(B)| + 1)$*-element subsequence consisting of the* $\bigoplus$ *-decomposition of* B.

LEMMA 2.33 ($\bigoplus$ -decomposition length for products). *Given flow graphs* A, B,

$$|\chi(A \otimes B)| = |\chi(A)| + |\Delta(A)| \cdot |\chi(B)|.$$

Proof. Consider each $\bigoplus$ -irreducible component $A^{(i)}$ in $\langle A\rangle$. If $A^{(i)} = F_1$ then by Observation 2.32, it contributes $(|\chi(B)| + 1)$ components in $\langle A \oplus B\rangle$. There are $|\Delta(A)|$ components in $\langle A\rangle$ which are isomorphic to F_1, so these together account for $|\Delta(A)| \cdot (|\chi(B)| + 1)$ components in $\langle A \otimes B\rangle$. The remaining $|\chi(A)| + 1 - |\Delta(A)|$ components in $\langle A\rangle$ (again by Observation 2.32) each contribute 1 component to $\langle A \otimes B\rangle$. It follows that the total number of components in $\langle A \otimes B\rangle$ is

$$|\chi(A)| + 1 - |\Delta(A)| + |\Delta(A)|(|\chi(B)| + 1) = \\ |\chi(A)| + |\Delta(A)||\chi(B)| + 1.$$

This shows that $|\chi(A \otimes B)| = |\chi(A)| + |\Delta(A)||\chi(B)|$ as desired. □

PROPOSITION 2.34 ($\bigoplus$ -irreducibility for products). *Let* $A \neq$ ① *be a flow graph. Then,* A *is* $\bigoplus$ *-irreducible iff for all flow graphs* B, $A \otimes B$ *is* $\bigoplus$ *-irreducible.*

Proof. [⇐] Taking $B =$ ①, we see that A is $\bigoplus$ -irreducible.

[⇒] If A is $\bigoplus$ -irreducible, then $\chi(A) = \emptyset$. Since $A \neq$ ①, it follows that $\Delta(A) = \emptyset$. So by Lemma 2.33, for any flow graph B, $\chi(A \otimes B) = \emptyset$. It follows that $A \otimes B$ is $\bigoplus$ -irreducible. □

DEFINITION 2.35. *A flow graph* A *is called* right-prime *if* A/B *exists only for* $B =$ ① *or* $B = A$. *Similarly a flow graph* A *is called* left-prime *if* $A\backslash B$ *exists only for* $B =$ ① *or* $B = A$.

Note that a natural number n is prime iff the flow graph F_n is prime.

LEMMA 2.36. *For all flow graphs* A, A *is right-prime iff* A *is left-prime.*

Proof. Suppose A is right-prime. Assume that $A\backslash B$ exists. We want to show that $B = F_1$ or $B = A$. Since $A\backslash B$ exists, it follows $A = B \otimes K$ for some K. Thus A/K exists, and since A is right-prime it follows that $K = F_1$ or $K = A$. If $K = A$ then $B = F_1$. If $K = F_1$, then $B = A$. □

2.3. Order Properties. In this section we explore the relationship between strong ordering by ⪯⃝ and weak ordering by ⩽⃝. While the two orders coincide on the graphical natural numbers, neither order is anti-symmetric on all of $\mathcal{F}$, and only ⪯⃝ is transitive. On the other hand, many of the laws that govern the relationship between $\leqslant$, $+$ and $\times$ in $\mathcal{N}$ continue to hold for ⩽⃝, $\bigoplus$ and $\otimes$ in $\mathcal{F}$, but these laws are violated under the ordering ⪯⃝.

LEMMA 2.37 (Strong Order Preservation). *For flow graphs* A, B, C, *if* A ⪯⃝ B *then*

$$(A \otimes C) ⪯⃝ (B \otimes C).$$

Proof. Let $A = (G_A, s_A, t_A)$, $B = (G_B, s_B, t_B)$. Since $A \circledless B$ there are graph embeddings $\phi_s : G_A \to G_B$ and $\phi_t : G_A \to G_B$ which satisfy $\phi_s(s_A) = s_B$ and $\phi_t(t_A) = t_B$. Define $\gamma_s : E[G_A] \times E[G_C] \to E[G_B] \times E[G_C]$ by

$$(e, f) \mapsto (\phi_s(e), f).$$

Then the composite map

$$\Phi_s^C : E[A \otimes C] \overset{\Lambda_{A,C}}{\longrightarrow} E[G_A] \times E[G_C] \overset{\gamma_s}{\longrightarrow} E[G_B] \times E[G_C] \overset{\Lambda^{-1}_{B,C}}{\longrightarrow} E[B \otimes C]$$

defines an embedding of $G_{A \otimes C} \to G_{B \otimes C}$ which takes $s_{A \otimes C}$ to $s_{B \otimes C}$. An analogous construction can be carried out to produce a map Φ_t^C which embeds $G_{A \otimes C} \to G_{B \otimes C}$ and sends $t_{A \otimes C}$ to $t_{B \otimes C}$. ∎

LEMMA 2.38 (Strong Order Violations). *There exist flow graphs A and B with $A \circledless B$ for which*

$$(i)\ \exists C_1 \in \mathcal{F} \text{ such that } (A \oplus C_1) \not\circledless (B \oplus C_1)$$
$$(ii)\ \exists C_2 \in \mathcal{F} \text{ such that } (C_2 \oplus A) \not\circledless (C_2 \oplus B)$$
$$(iii)\ \exists C_3 \in \mathcal{F} \text{ such that } (C_3 \otimes A) \not\circledless (C_3 \otimes B).$$

Proof. See Figure 2.4. ∎

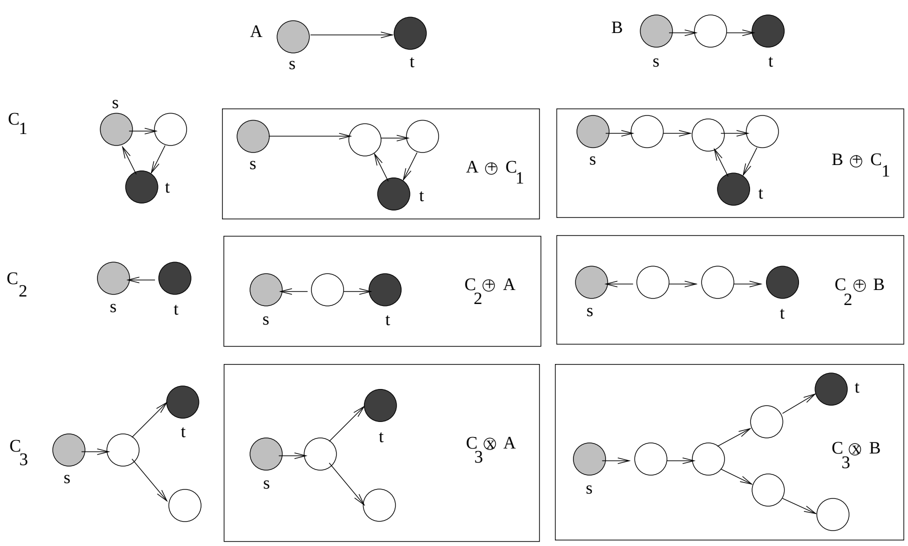

FIG. 2.4. *Strong order violations: (i). $(A \oplus C_1) \not\circledless (B \oplus C_1)$, (ii). $(C_2 \oplus A) \not\circledless (C_2 \oplus B)$, and (iii). $(C_3 \otimes A) \not\circledless (C_3 \otimes B)$.*

We consider possible anti-symmetry of $\circledless$. Suppose $A \circledless B$ and $B \circledless A$. There is a graph embedding $\phi_s : G_A \to G_B$ which satisfies $\phi_s(s_A) = s_B$. Hence $|V[G_A]| = |V[\phi_s(G_A)]| \leqslant |V[G_B]|$ and $|E[G_A]| = |E[\phi_s(G_A)]| \leqslant |E[G_B]|$. Since $B \circledless A$, there is a graph embedding $\psi_s : G_B \to G_A$ which satisfies $\psi_s(s_B) = s_A$. So $|V[G_B]| = |V[\psi_s(G_B)]| \leqslant |V[G_A]|$ and $|E[G_B]| = |E[\psi_s(G_B)]| \leqslant |E[G_A]|$. It follows that ϕ_s is actually an isomorphism from G_A to G_B satisfying $\phi_s(s_A) = s_B$. A similar argument shows that there is an isomorphism ϕ_t from G_A to G_B satisfying $\phi_t(t_A) = t_B$. To conclude that $A = B$ requires a single flow graph isomorphism π from A to B, satisfying *both* $\pi(s_A) = s_B$ and $\pi(t_A) = t_B$. Indeed in some cases, no such isomorphism may exist.

EXAMPLE 2.39. *Let G_A be a directed cycle of length 4, and take s_A, t_A to be any two vertices in $V[G_A]$ that are distance 2 apart. Put G_B isomorphic to G_A, taking s_B, t_B to be two vertices in $V[G_B]$ that are distance 1 apart. Then it is easy to verify that $(G_A, s_A, t_A) = A \olessthan B = (G_B, s_B, t_B)$ and $B \olessthan A$. Clearly, however, $A \neq B$ as flow graphs (see Figure 2.5).*

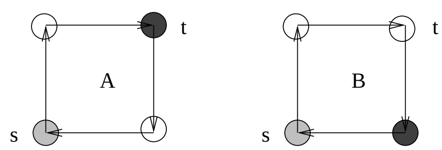

FIG. 2.5. *An example which demonstrates that the strong order is not antisymmetric.*

The previous example proves the next lemma.

LEMMA 2.40 (Non-antisymmetry of strong order $\olessthan$). *There exist flow graphs A and B for which*

$$A \olessthan B \text{ and } B \olessthan A \text{ but } A \neq B.$$

LEMMA 2.41 (Transitivity of strong order $\olessthan$). *For all flow graphs A, B, C*

$$A \olessthan B \text{ and } B \olessthan C \text{ implies } A \olessthan C.$$

Proof. $A \olessthan B$: i.e. there are graph embeddings $\phi_s : G_A \to G_B$ and $\phi_t : G_A \to G_B$ which satisfy $\phi_s(s_A) = s_B$ and $\phi_t(t_A) = t_B$. $B \olessthan C$: i.e. there are graph embeddings $\theta_s : G_B \to G_C$ and $\theta_t : G_B \to G_C$ which satisfy $\theta_s(s_B) = s_C$ and $\theta_t(t_B) = t_C$. We want to show $A \olessthan C$: i.e. there are graph embeddings $\alpha_s : G_A \to G_C$ and $\alpha_t : G_A \to G_C$ which satisfy $\alpha_s(s_A) = s_C$ and $\alpha_t(t_A) = t_C$. Put $\alpha_s = \theta_s \circ \phi_s$ and $\alpha_t = \theta_t \circ \phi_t$. □

LEMMA 2.42 (Weak Order Preservation). *For flow graphs A, B, C, if $A \circledleq B$ then*

$$(i)\ (A \oplus C) \circledleq (B \oplus C)$$
$$(ii)\ (C \oplus A) \circledleq (C \oplus B)$$
$$(iii)\ (A \otimes C) \circledleq (B \otimes C).$$

Proof. Let $A = (G_A, s_A, t_A), B = (G_B, s_B, t_B)$ and $C = (G_C, s_C, t_C)$ be given. $A \circledleq B$ implies that there exists an (s_A, t_A)-splitting (H_1, H_2) of G_A and graph embeddings

$$\phi_1 : H_1 \to G_B$$
$$\phi_2 : H_2 \to G_B$$

satisfy $\phi_1(s_A) = s_B$ and $\phi_2(t_A) = t_B$ and $\phi_1(E[H_1]) \cap \phi_2(E[H_2]) = \emptyset$.

(i). Put $K_1 = H_1$ and define K_2 to be the graph obtained by gluing H_2 and G_C such that t_A is identified with s_C. Now define $\Phi_1 = \phi_1 : H_1 \to G_B \bigoplus_{t_B \approx s_C} G_C$, $\Phi_2|_{H_2} = \phi_2 : H_2 \to G_B \bigoplus_{t_B \approx s_C} G_C$ and $\Phi_2|_{G_C} : G_C \to G_B \bigoplus_{t_B \approx s_C} G_C$. Then (K_1, K_2) is an (s_A, t_C)-splitting of $G_A \bigoplus_{t_A \approx s_C} G_C$ and

$$\Phi_1 : K_1 \to G_B \bigoplus_{t_B \approx s_C} G_C,$$
$$\Phi_2 : K_2 \to G_B \bigoplus_{t_B \approx s_C} G_C$$

are graph embeddings satisfying $\Phi_1(s_A) = s_B$ and $\Phi_2(t_C) = t_C$.

(ii). We defineL_1 to be the graph obtained by gluing G_C and H_1 such that t_C is identified with s_A and we put $L_2 = H_2$.

$$\theta_1|_{G_C} : G_C \to G_C \bigoplus_{t_C \approx s_B} G_B$$
$$\theta_1|_{H_1} : H_1 \to G_C \bigoplus_{t_C \approx s_B} G_B$$
$$\theta_2 = \phi_2 : H_2 \to G_C \bigoplus_{t_C \approx s_B} G_B.$$

Then (L_1, L_2) is an (s_C, t_A)-splitting of $G_C \bigoplus_{t_C \approx s_A} G_A$ and

$$\theta_1 : L_1 \to G_C \bigoplus_{t_C \approx s_B} G_B,$$
$$\theta_2 : L_2 \to G_C \bigoplus_{t_C \approx s_B} G_B$$

are graph embeddings satisfying $\theta_1(s_C) = s_C$ and $\theta_2(t_A) = t_B$.

(iii). Put

$$M_1 = H_1 \bigotimes C$$
$$M_2 = H_2 \bigotimes C.$$

Since $E[H_1] \cup E[H_2] = E[G_A]$ and $E[H_1]) \cap E[H_2] = \emptyset$, it follows that $E[H_1 \bigotimes C] \cup E[H_2 \bigotimes C] = E[G_A \bigotimes C]$ and $E[H_1 \bigotimes C] \cap E[H_2 \bigotimes C] = \emptyset$. Thus M_1, M_2 are an (s,t)-splitting of $G_A \bigotimes C$. Now take $\beta_1 : E[M_1] \to E[G_B \bigotimes G_C]$ and $\beta_2 : E[M_2] \to E[G_B \bigotimes G_C]$ defined by

$$\beta_1 : (e, f) \mapsto (\phi_1 e, f)$$
$$\beta_2 : (e', f) \mapsto (\phi_2 e', f)$$

for $e \in E[H_1]$, $e' \in E[H_2]$ and $f \in C$. The injectivity of β_1 and β_2 follows immediately from injectivity of ϕ_1 and ϕ_2. Since $\phi_1(E[H_1]) \cap \phi_2(E[H_2]) = \emptyset$, it follows that $\beta_1(E[H_1 \bigotimes C]) \cap \beta_2(E[H_2 \bigotimes C]) = \emptyset$. Since $\phi_1(s_A) = s_B$ and $\phi_2(t_A) = t_B$, it follows that $\beta_1(s_{A \bigotimes C}) = s_{B \bigotimes C}$ and $\beta_2(t_{A \bigotimes C}) = t_{B \bigotimes C}$. Thus, the maps β_1 and β_2 demonstrate $A \bigotimes C \lessdot B \bigotimes C$. □

LEMMA 2.43 (Weak Order Violations). *There exist flow graphs A, B, C for which $A \lessdot B$ but*

$$(C \bigotimes A) \not\lessdot (C \bigotimes B).$$

Proof. See Figure 2.6. □

LEMMA 2.44 (Non-transitive weak order $\lessdot$). *There exist flow graphs A, B, C*

$$A \lessdot B \text{ and } B \lessdot C \text{ but } A \not\lessdot C.$$

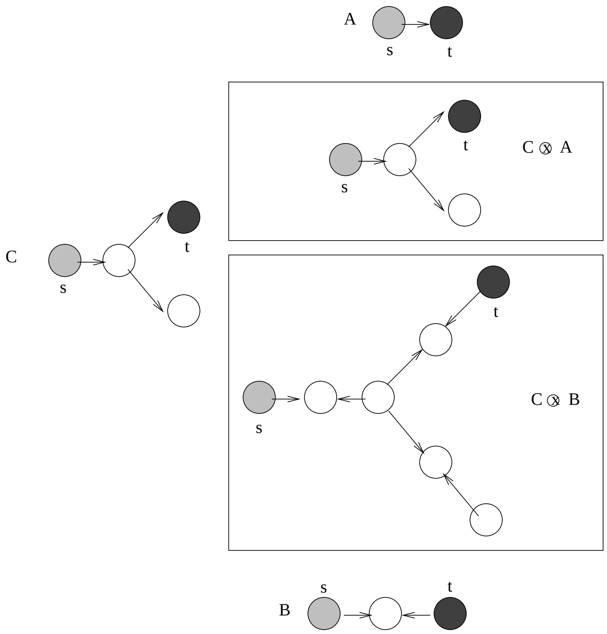

FIG. 2.6. *An example which demonstrates weak order violation:* $(C \otimes A) \not\circledless (C \otimes B)$.

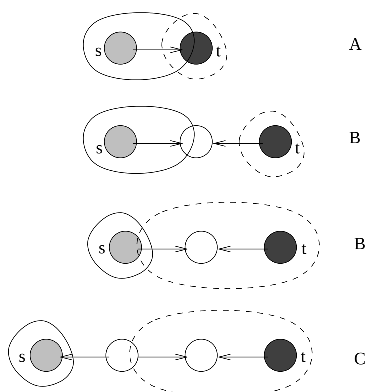

FIG. 2.7. *An example which demonstrates that the weak order is not transitive.*

Proof. See Figure 2.7. □

Since strong order implies weak order, Lemma 2.40 and the example in Figure 2.5 immediately yield:

LEMMA 2.45 (Non-antisymmetry of weak order $\circledless$). *There exist flow graphs A and B for which*

$$B \circledless A \text{ and } A \circledless B \text{ but } A \neq B$$

3. Conclusions and Future Work. As we have seen, strikingly many theorems that are true in $\mathcal{N}$ continue to hold in $\mathcal{F}$, though some fail. Our future research program will proceed on two tracks.

Informally, for each "classical" theorem ϕ in $TA \backslash Th(\mathcal{F})$:

(1) We shall consider the structure of maximal subsets X_ϕ which have the property that the submodel $\mathcal{X}_\phi \stackrel{def}{=} (\mathcal{F}|_{X_\phi}) \models \phi$. Of particular interest are sets X_ϕ which properly contain $i(\mathbb{N})$.

(2) We shall describe a corresponding theorem ϕ' in $Th(\mathcal{F})$, such that $\phi' \equiv \phi$ when restricted to $i(\mathbb{N})$.

Examples of specific questions include:

i. Characterize flow graph pairs for which antisymmetry of strong order holds.

ii. Characterize $\otimes$ -commuting pairs, i.e. under what conditions on flow graphs A and B does $A \otimes B = B \otimes A$?

iii. Does $A \otimes B = A \otimes C$ imply $B = C$? Does $B \otimes A = C \otimes A$ imply $B = C$? In other words, does $\otimes$ satisfy a left/right cancellation law?

iv. *Graph Prime Factorization Conjecture.* Every flow graph is uniquely expressible (up to some well-defined reordering) as the product of prime flow graphs.

v. Describe solution sets (in $\mathcal{F}$) for one-variable equations having the form $p(x) = q(x)$, where p and q are polynomials with coefficients from $\mathcal{F}$.

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REFERENCES

[1] K. R. Bhutani and B. Khan, *A metric representing dilation between connected simple graphs of a given order.* Submitted to Discrete Applied Mathematics.

[2] ———, *Distance between graphs using graph labelings*, Ars Combinatoria, (2003). Accepted for publication.

[3] ———, *A metric on the class of connected simple graphs of given order*, Aequationes Mathematicae, 66 (2003), pp. 232–240.

[4] W. Imrich and S. Klavzar, *Product Graphs, Structure and Recognition*, John Wiley & Sons Inc., 2000.

[5] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2nd ed., 2001.