Lower Bounds on the Order of $\psi(x) - x$

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Abstract

In 1914, a short note titled "Sur la distribution des nombres premiers" appeared in the *Comptes Rendus* of the Acadamie Sciences. In it, J. E. Littlewood sketched proofs of two new results:

$$\psi(x) - x = \Omega_R(\sqrt{x} \log\log\log x)$$
$$\psi(x) - x = \Omega_L(\sqrt{x} \log\log\log x)$$

Some years later, in 1931, a joint paper by G. H. Hardy and Littlewood titled "The Riemann-Zeta function and the theory of distribution of primes" was published in Acta Mathematica. Among the many results presented in it was a more complete exposition of Littlewood's original 1914 theorems concerning asymptotic lower bounds on $\psi(x) - x$.

In this paper, we shall present a detailed account of the original proofs of these theorems.

1 Introduction

The function $\pi(x)$ is defined to be the number of prime integers less than or equal to x. In 1863, Gauss published results in which suggested $1/\log x$ as a reasonable approximation to the average density of the distribution of primes, from which it followed that

$$\pi(x) \approx \int_2^x \frac{du}{\log u}$$

For convenience many researchers have focused instead on the "logarithmic integral" function

$$Li(x) \overset{\text{def}}{=} \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} \frac{du}{\log u} + \int_{1+\epsilon}^x \frac{du}{\log u} \right)$$

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$\mathrm{Li}(x)$ differs from Gauss' estimate of $\pi(x)$ by the constant $\mathrm{Li}(2) = 1.04\cdots$, but the constant additive difference is irrelevant to questions of asymptotic density.

The study of how well $\mathrm{Li}(x)$ approximates $\pi(x)$ is a central question in the study of the distribution of primes. In 1848 Chebychev showed that if $\lim \pi(x)/(x/\log x)$ exists then it must be 1. After Chebyshev's work, progress on characterizing the distribution of primes fell into one of two broad categories. On the one hand were the "O" results, or upper bounds on the magnitude of $\pi(x){-}\mathrm{Li}(x)$, such as the Prime Number Theorem proved by Hadamard and de la Vallée Poussin in 1896, which showed that $\pi(x){-}\mathrm{Li}(x) = O(\frac{x}{\log x})$. On the other hand were theorems exhibiting lower bounds on the magnitude of $\pi(x){-}\mathrm{Li}(x)$, or the so-called "Ω" theorems. The first results of this latter type were proved by Littlewood in 1914, and are the subject of this exposition. Specifically, in 1914 Littlewood showed in [8,3] that

$$\psi(x) - x = \Omega_R(\sqrt{x}\log\log\log x)$$

$$\psi(x) - x = \Omega_L(\sqrt{x}\log\log\log x)$$

The above expressions are shorthand for the assertion that $\exists K > 0$, such that $\forall N > 0$, $\exists x_L, x_R \geq N$, satisfying

$$\psi(x_R) - x_R > K\sqrt{x_R}\log\log\log x_R$$

$$\psi(x_L) - x_L < -K\sqrt{x_L}\log\log\log x_L$$

We begin by briefly describing three crucial ideas, all known to Littlewood in 1914, that were central to the successful execution of his proof. These are: (i) the Explicit Formula for $\psi(x) - x$, (ii) the von Mangoldt Formula, and (iii) the Phragmén-Lindelöf Theorem.

1.1 $\pi(x){-}\mathrm{Li}(x)$ versus $\psi(x) - x$

Chebychev was the first to introduce the function

$$\psi(x) \overset{\text{def}}{=} \sum_{p^m \leq x} \log p$$

where the sum on the right-hand side extends over primes p and positive integers m that satisfy $p^m \leq x$. In particular, Chebychev was the first to show that the quantities

$$\pi(x)/\frac{x}{\log x}\ ,\ \psi(x)/x$$

have the same limits as $x \to +\infty$, provided the limits exist. Subsequent work on $\pi(x){-}\mathrm{Li}(x)$, being motivated by a quest for a proof of "the prime number theorem" often explored the quantity $\psi(x) - x$. Conclusions regarding $\psi(x) - x$

could then be then easily made to imply the corresponding conclusions about $\pi(x)-\mathrm{Li}(x)$. Littlewood also follows this approach. (In this exposition we shall not address how to transform Littlewood's results on $\psi(x) - x$ to conclusions about $\pi(x)-\mathrm{Li}(x)$, as this involves fairly standard partial summation techniques.)

1.2 An Explicit Formula for $\psi(x) - x$

A quantity closely related to $\psi(x)$ is

$$\psi_0(x) \stackrel{\text{def}}{=} \frac{\psi(x+0) - \psi(x-0)}{2}$$

It is immediate from the definitions, that $\psi_0(x)$ differs from $\psi(x)$ only when $x = p^m$ is a prime power, and that then the difference between the two quantities is $\frac{\log p}{2} = O(\log x)$.

The first key ingredient in Littlewood's proof is the "Explicit Formula for $\psi_0(x)-x$", which asserts that for $x > 1$,

$$\psi_0(x) - x \quad = \quad -2\sum_{\rho} \frac{x^{\rho}}{\rho} + \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) \tag{1}$$

where

$$\sum_{\rho} \frac{x^{\rho}}{\rho} \stackrel{\text{def}}{=} \lim_{T\to\infty} \sum_{|\gamma_n|\leq T} \frac{x^{\rho}}{\rho}$$

is the limit of the summation over all zeros $\rho = (\xi + i\gamma_n)$ of the Riemann-Zeta function. For the derivation of this formula, the reader is referred to the treatment by Patterson in [9], pages 43-46.

1.3 The von Mangoldt Formula

The second key ingredient in Littlewood's proof is the formula for $N(T)$, the number of zeros $\rho = (\xi + i\gamma_n)$ of the Riemann-Zeta function in the box $0 \leq \xi \leq 1$, $0 \leq \gamma_n \leq T$. Originally stated without proof by Riemann in 1859,

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T) \tag{2}$$

the formula was proved by von Mangoldt in 1895. The details of its proof are presented succinctly by Ingham in [6], pages 68-73.

1.4 The Phragmén-Lindelöf Theorem

The third key ingredient in Littlewood's proof is an extension of the maximum-modulus theorem of complex variables given by Phragmén and Lindelöf.

In its original form, the theorem states that

Theorem (Phragmén-Lindelöf) Given f an analytic function of $re^{i\theta}$ that is regular in the region D of the complex plane between (and including) two straight lines that make an angle π/α at the origin. Suppose we are given that $|F(z)| \leq M$ on the two lines and that as $r \to \infty$, $f(z) = O(e^{r^\beta})$ where $\beta < \alpha$ uniformly in the interior of D. *Then*, $|F(z)| \leq M$ throughout the region D.

The region D may be transformed into a differently-shaped region, e.g. a vertical strip, and an analogous assertion obtained for such regions.

A complete introduction to the Phragmén-Lindelöf Theorem and its consequences is given by Titchmarsh in [10], pages 176-181.

2 An Outline of the Proof

Before presenting the full details of Littlewood's proof, we briefly outline the arc of his argument:

Littlewood begins by noticing that a "formula to be found in Landau's *Handbuch*" implies

$$\frac{\psi(x) - x}{\sqrt{x}} = -2\sum_{n=1}^{\infty} \frac{\sin \gamma_n \log x}{\gamma_n} + O(1)$$

which transforms the investigation of $\psi(x)-x$ into one of analyzing $-2\sum_{n=1}^{\infty} \frac{\sin \gamma_n \log x}{\gamma_n}$.

Littlewood notes, however, that

$$\sum_{n=1}^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n} = \Im F(i\eta)$$

where $z = \xi + i\eta$ and

$$F(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{e^{-\gamma_n z}}{\gamma_n}$$

First, he approximates F in the complex plane by using von Mangoldt's formula to show that near 0 along the line $\Re(z) = \Im(z)$

$$-\Im F(\xi + i\xi) = \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \xi}{\gamma_n} \approx \frac{1}{8} \log \frac{1}{\xi} \tag{3}$$

Then he obtains an inequality for the tail of the summation defining $F(z)$, when z is near 0 along the line $\Re(z) = \Im(z)$. In particular, he shows that for each $\xi > 0$, $\exists$ constant a, such that

$$\sum_{\gamma_n \xi > a} \frac{e^{-\gamma_n \xi}}{\gamma_n} < \frac{1}{32} \log \frac{1}{\xi} \tag{4}$$

Moving away from the line $\Re(z) = \Im(z)$, the observations (3) and (4) enable him to harness a Dirichlet theorem to conclude that for every small $\xi > 0$, there exists a corresponding value of $\eta > 0$ such that

$$-\Im F(\xi + i\eta) > \frac{1}{24} \log \frac{1}{\xi} \tag{5}$$

The numeric lower bound (5) translates into a Ω lower bound,

$$-\Im F(\xi + i\eta) = \Omega_{L|R}(\log \log \eta) \tag{6}$$

Now Littlewood considers

$$-\Im F(i\eta) \overset{\text{def}}{=} \lim_{\xi \to 0} -\Im F(\xi + i\eta)$$

and by a modification of the Phragmén-Lindelöf Theorem, he shows that the Ω bound (6) on $-\Im F(\xi + i\eta)$ continues to holds when $\xi \to 0$. In other words,

$$-\Im F(i\eta) = \Omega_R(\log \log \eta) \ \ , \ \ -\Im F(i\eta) = \Omega_L(\log \log \eta) \tag{7}$$

Finally, the Ω bounds (7) on $-\Im F(i\eta)$ imply Littlewood's ultimate conclusion, the existence of Ω bounds on $\psi(x) - x$.

$$\psi(x) - x = \Omega_R(\sqrt{x} \log \log \log x) \tag{8}$$

$$\psi(x) - x = \Omega_L(\sqrt{x} \log \log \log x) \tag{9}$$

3 Littlewood's Proof

3.1 A Formula in Landau's *Handbuch*

Starting with the Explicit Formula, when $x > 1$,

$$\psi_0(x) - x = -2\sum_\rho \frac{x^\rho}{\rho} + \underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{\log 2\pi} \underbrace{-\frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)}_{O(1)}$$

From which we can obtain a sufficient approximation for $\psi(x) - x$

$$\psi(x) - x = -2\sum_\rho \frac{x^\rho}{\rho} + \log x + O(1)$$

We fix $T > x^2$ arbitrary, and split the summation

$$\psi(x) - x = -2\sum_{\gamma_n < T} \frac{x^\rho}{\rho} \underbrace{-2\sum_{\gamma_n >= T} \frac{x^\rho}{\rho}}_{O(\sqrt{x})} + \log x + O(1)$$

Now assuming the Riemann Hypothesis[1], we take $\rho = \frac{1}{2} + i\gamma_n$, from which we deduce

$$\begin{aligned}
\psi(x) - x &= -2\sum_{\gamma_n < T} \frac{x^{\frac{1}{2}+i\gamma_n}}{\frac{1}{2} + i\gamma_n} + O(\sqrt{x}) \\
\frac{\psi(x) - x}{\sqrt{x}} &= -2\sum_{\gamma_n < T} \frac{x^{i\gamma_n}}{\frac{1}{2} + i\gamma_n} + O(1) \\
\frac{\psi(x) - x}{\sqrt{x}} &= -2\sum_{\gamma_n < T} \frac{x^{i\gamma_n}}{i\gamma_n} + \sum_{\gamma_n < T}\left(\frac{x^{i\gamma_n}}{i\gamma_n} - \frac{x^{i\gamma_n}}{\frac{1}{2} + i\gamma_n}\right) + O(1)
\end{aligned}$$

But

$$x^{i\gamma_n} = \cos(\gamma_n \log x) + i\sin(\gamma_n \log x) = O(1)$$

so

$$\frac{\psi(x) - x}{\sqrt{x}} = -2\sum_{\gamma_n < T} \frac{x^{i\gamma_n}}{i\gamma_n} + \sum_{\gamma_n < T}\left(\frac{1}{i\gamma_n} - \frac{1}{\frac{1}{2} + i\gamma_n}\right) + O(1)$$

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[1] If the Riemann Hypothesis is taken to be false then much stronger statements than these results of Littlewood's would be true. This line of argument is described in detail by Landau in [7], pages 712-.

$$
\begin{aligned}
\underbrace{\frac{\psi(x)-x}{\sqrt{x}}}_{\text{real}} &= -2\sum_{\gamma_n<T}\frac{x^{i\gamma_n}}{i\gamma_n} + \underbrace{O\sum_{\gamma_n<T}\left(\frac{1}{i\gamma_n+2\gamma_n^2}\right)}_{O(1)} + O(1)\\
\frac{\psi(x)-x}{\sqrt{x}} &= -2\ \Re\left(\sum_{\gamma_n<T}\frac{x^{i\gamma_n}}{i\gamma_n} + O(1)\right)\\
\frac{\psi(x)-x}{\sqrt{x}} &= -2\sum_{|\gamma_n|<T}\frac{\sin(\gamma_n x)}{\gamma_n} + O(1) \qquad (10)
\end{aligned}
$$

Littlewood begins his exposition by noting that (10) is “a corollary of a known formula to be found [on pages 387 and 351] in Landau’s *Handbuch*”.

Since the series on the right of (10) is known to be convergent, he takes the limit as $T \to +\infty$, to obtaining the equation

$$\frac{\psi(x)-x}{\sqrt{x}} = -2\sum_{n=1}^{\infty}\frac{\sin(\gamma_n \log x)}{\gamma_n} + O(1)$$

Littlewood’s final goal of showing (8) and (9) then reduces proving

$$\sum_{n=1}^{\infty}\frac{\sin(\gamma_n \log x)}{\gamma_n} = \Omega_{R|L}(\log\log\log x) \qquad (11)$$

which is equivalent (by taking $\eta = \log x$) to showing that

$$\sum_{n=1}^{\infty}\frac{\sin(\gamma_n \eta)}{\gamma_n} = \Omega_{R|L}(\log\log\eta) \qquad (12)$$

3.2 The Function $F(z) = \sum_{n=1}^{\infty}\frac{e^{-\gamma_n z}}{\gamma_n}$

Rather than perform computations using the somewhat messy function of one variable

$$\sum_{n=1}^{\infty}\frac{\sin\gamma_n\eta}{\gamma_n}$$

Littlewood considers the complex-valued function

$$F(z) \overset{\text{def}}{=} \sum_{n=1}^{\infty}\frac{e^{-\gamma_n z}}{\gamma_n}$$

since when $z = i\eta$ is purely imaginary,

$$F(i\eta) = \sum_{n=1}^{\infty} \frac{e^{-\gamma_n (i\eta)}}{\gamma_n}$$

and hence,

$$-\Im F(i\eta) = \sum_{n=1}^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n}$$

which is the expression (12) that he is interested in. Thus Littlewood's task is transformed into showing that

$$-\Im F(i\eta) = \Omega_{R|L}(\log\log \eta) \tag{13}$$

3.3 Analyzing F Near 0, on the Line $\Re(z) = \Im(z)$

To prepare for an eventual investigation of $-\Im F(i\eta)$, Littlewood begins by considering the behavior of $-\Im F(z)$ in the complex plane along the line $\Re(z) = \Im(z)$. Note that

$$-\Im F(\xi + i\xi) = \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \xi}{\gamma_n}$$

The right-hand side involves summing over all zeros of the Riemann-Zeta function, and one may approximate the sum by appealing to estimates implicit in the von Mangoldt formula (2).

Let us take $z(T)$ to be

$$z(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T.$$

Clearly $z(T) \approx N(T)$ is monotonic, so define $g(u)$ to be the function inverse to $z(T)$, i.e.

$$z(T) = u \quad \leftrightarrow \quad g(u) = T$$

Intuitively, $g(n)$ approximates γ_n. More precisely, because $|N(T) - z(T)| = O(\log T)$, it follows that

$$\gamma_n = g(n) + O(1)$$

Since z is monotonic, so is g. The derivative of $g = z^{-1}$ is

$$\begin{aligned} g'(u) &= \frac{1}{z'(g(u))} \\ &= \frac{1}{\frac{1}{2\pi} \log g(u) + \frac{1}{2\pi} - \frac{1+\log 2\pi}{2\pi}} \end{aligned}$$

$$\begin{aligned} &= \frac{2\pi}{\log g(u) - \log 2\pi} \\ &= \frac{2\pi}{\log\left(\frac{g(u)}{2\pi}\right)} \end{aligned} \tag{14}$$

Consider just one term in the summation $\sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi}\sin \gamma_n \xi}{\gamma_n}$, and compare it to what is obtained by using the approximation $g(u)$, $n \leq u \leq n+1$, in place of γ_n.

$$\begin{aligned}
\frac{e^{-(\xi+i\xi)g(u)}}{g(u)} &= \frac{e^{-(\xi+i\xi)(\gamma_n+O(1))}}{g(u)-c_1} \cdot \frac{g(u)-c_1}{g(u)} \\
&= \frac{e^{-(\xi+i\xi)(\gamma_n+O(1))}}{\gamma_n} \cdot \left(1+\frac{c_1}{g(u)}\right) \\
&= \frac{e^{-(\xi+i\xi)(\gamma_n+O(1))}}{\gamma_n} \cdot \left(1+\frac{c_1}{g(u)}\right) \\
&= \frac{e^{-(\xi+i\xi)(\gamma_n+O(1))}}{\gamma_n} \cdot \left\{1+O\left(\frac{1}{g(u)}\right)\right\} \\
&= \frac{e^{-(\xi+i\xi)\gamma_n+O(1)}}{\gamma_n} \left\{1+O\left(\frac{\log n}{n}\right)\right\} \\
&= \frac{e^{-(\xi+i\xi)(\gamma_n+c_2)}}{\gamma_n} \left\{1+O\left(\frac{\log n}{n}\right)\right\} \\
&= \frac{e^{-(\xi+i\xi)\gamma_n+(\xi c_2+i\xi c_2)}}{\gamma_n} \left\{1+O\left(\frac{\log n}{n}\right)\right\} \\
&= \underbrace{e^{\xi c_2}}_{\approx\, 1+\xi c_2 \,=\, O(1+\xi)} \underbrace{(\cos \xi c_2 + i \sin \xi c_2)}_{O(1)} \frac{e^{-(\xi+i\xi)\gamma_n}}{\gamma_n} \left\{1+O\left(\frac{\log n}{n}\right)\right\}
\end{aligned}$$

Thus for $\xi \approx 0$,

$$\frac{e^{-(\xi+i\xi)g(u)}}{g(u)} = \frac{e^{-(\xi+i\xi)\gamma_n}}{\gamma_n}\left\{1+O(\xi)+O\left(\frac{\log n}{n}\right)\right\}$$

Having gotten some control over the degree to which $g(u)$, $n \leq u \leq n+1$ can be used to approximate γ_n, we return to the original summation defining F, with the intention of using the former in place of the latter. Let u_0 be any fixed positive number. Then,

$$-\Im F(\xi+i\xi) \quad = \quad \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi}\sin \gamma_n \xi}{\gamma_n}$$

$$\begin{aligned}
&= \int_{u_0}^{+\infty} \frac{e^{-(\xi+i\xi)g(u)}}{g(u)} du + O\int_{u_0}^{+\infty} \frac{e^{-(\xi+i\xi)g(u)}}{g(u)}\left(\xi + \frac{\log u}{u}\right) du + O(1) \\
&= \int_{u_0}^{+\infty} \frac{e^{-(\xi+i\xi)g(u)}}{g(u)} du \\
&\quad +O\int_{u_0}^{+\infty} \underbrace{(\cos\xi + i\sin\xi)}_{O(1)} \frac{e^{-\xi g(u)}}{g(u)}\left(\xi + \frac{\log u}{u}\right) du + O(1) \\
&= \int_{u_0}^{+\infty} \frac{e^{-(\xi+i\xi)g(u)}}{g(u)} du + O\int_{u_0}^{+\infty} \frac{e^{-(\xi+i\xi)g(u)}}{g(u)}\left(\xi + \frac{\log u}{u}\right) du + O(1)
\end{aligned}$$

Changing the variable of integration and defining $g_0 = g(u_0)$, we conclude

$$-\Im F(\xi+i\xi) = \int_{g_0}^{+\infty} \frac{e^{-(\xi+i\xi)g}}{g}\frac{dg}{g'(u)} + O\left(\xi \underbrace{\int_{g_0}^{+\infty} \frac{e^{-\xi g}}{g}\frac{dg}{g'(u)}}_{A}\right) + O(1)$$

By (14),

$$\begin{aligned}
\frac{dg}{g'(u)} &= \frac{\log\left(\frac{g(u)}{2\pi}\right)}{2\pi} dg \\
&= \frac{\log g(u) - \log 2\pi}{2\pi} dg \\
&= O(\log g(u)) dg
\end{aligned}$$

So term A evaluates to

$$\begin{aligned}
\int_{g_0}^{+\infty} \frac{e^{-\xi g}}{g}\frac{dg}{g'(u)} &= O\int_{g_0}^{+\infty} \frac{e^{-\xi g}\log g}{g} dg \\
&= O\left(\log\frac{1}{\xi}\right)^2
\end{aligned}$$

Reducing, we obtain

$$F(\xi+i\xi) \quad = \quad \frac{1}{2\pi}\int_{g_0}^{+\infty} \frac{e^{-(\xi+i\xi)g}}{g}(\log g - 2\pi)\; dg + O(\underbrace{\xi\left(\log\frac{1}{\xi}\right)^2}_{\text{real}}) + O(1)$$

$$\begin{aligned}\text{and so } \Im F(\xi+i\xi) &= \frac{1}{2\pi}\int_{g_0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g}(\log g - \log 2\pi)\, dg + O(1) \\ &= \underbrace{\frac{1}{2\pi}\int_{g_0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g}\log g\, dg}_{B} - \underbrace{\frac{\log 2\pi}{2\pi}\int_{g_0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g} dg}_{C} + O(1)\end{aligned}$$

Now term B, having positive sign, may be sufficiently approximated by noting

$$\begin{aligned}\frac{1}{2\pi}\int_{g_0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g}\log g\, dg &\leq \frac{1}{2\pi}\int_{0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g}\log g\, dg \\ \text{thus } \Im B &= \frac{1}{2\pi}\int_{0}^{+\infty}\frac{e^{-\xi g}\sin(\xi g)\cdot\log g}{g}\, dg + O(1)\end{aligned}$$

Changing variables by defining $\omega = \xi g$, we get that

$$\begin{aligned}\Im B &= \frac{1}{2\pi}\int_{0}^{+\infty}\frac{e^{-\omega}\sin(\omega)\cdot(\log\omega - \log\xi)}{\omega}\, d\omega + O(1) \\ &= \frac{1}{8}\log\left(\frac{1}{\xi}\right) + O(1)\end{aligned}$$

Now $u_0 > 0$ implies $g_0 > 0$. Hence the integral defined by the term C, evaluates to a constant.

$$\frac{\log 2\pi}{2\pi}\int_{g_0}^{+\infty}\frac{e^{-(\xi+i\xi)g}}{g} dg = O(1)$$

Thus we conclude

$$-\Im F(\xi+i\xi) = \frac{1}{8}\log\frac{1}{\xi} + O(1) \tag{15}$$

3.4 The Tail of the Summation Defining F

Next, Littlewood considers the tails of the summation

$$\sum_n \frac{e^{-\gamma_n\xi}}{\gamma_n}$$

As a consequence of the von Mangoldt formula, the number of roots of the Riemann-Zeta function that lie between ν and $\nu+1$ is $O(\log\nu)$.

Thus

$$\begin{aligned}
\sum_{\gamma_n \xi > a} \frac{e^{-\gamma_n \xi}}{\gamma_n} &= O\left(\sum_{\nu > \frac{a}{\xi} - 1} \frac{e^{-\nu \xi}}{\nu} \log \nu \right) \\
&= O\left(\int_{\frac{a}{\xi}}^{+\infty} \frac{e^{-u\xi}}{u} \log u \, du \right) \\
&= O\left\{ \int_a^{+\infty} \frac{e^{-w} \log w}{w} \, dw + \log\left(\frac{1}{\xi}\right) \int_a^{+\infty} \frac{e^{-w} \log w}{w} \, dw \right\}
\end{aligned}$$

Computation for any constant $a > 0$ shows that

$$\begin{aligned}
\int_a^{+\infty} \frac{e^{-w} \log w}{w} \, dw &= \int_a^1 \frac{e^{-w} \log w}{w} \, dw + \int_1^{+\infty} \frac{e^{-w} \log w}{w} \, dw \\
&\leq \underbrace{\int_a^1 \frac{e^{-w} \log w}{w} \, dw}_{D} + \underbrace{\sup_{x \in [1,\infty)} \frac{\log w}{w}}_{O(1)} \cdot \underbrace{\int_1^{+\infty} e^{-w} \, dw}_{O(1)}
\end{aligned}$$

And since D is the integral of a monotonically decreasing function, it follows that

$$0 \geq D \geq (1-a) \frac{e^{-a} \log a}{a}$$

For any $a > 0$ constant,

$$(1-a) \frac{e^{-a} \log a}{a} = O(1)$$

and thus

$$\int_a^{+\infty} \frac{e^{-w} \log w}{w} \, dw = O(1)$$

permitting us to conclude that

$$\sum_{\gamma_n \xi > a} \frac{e^{-\gamma_n \xi}}{\gamma_n} = O \log\left(\frac{1}{\xi}\right)$$

And an explicit calculation of the constant hidden in the O of the above equation reveals the following assertion to be true

$$\sum_{\gamma_n \xi > a} \frac{e^{-\gamma_n \xi}}{\gamma_n} \leq \frac{1}{32} \log\left(\frac{1}{\xi}\right)$$

3.5 A Dirichlet Theorem

If x is any real, define $\tilde{x}$ to be the number that differs from x by an integer, and satisfies the inequalities $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

Theorem. Given any positive numbers: ρ_0 (large), ξ (small), and N (integral), there exists a ρ such that

$$\rho_0 \leq \rho \leq \rho_0 \left(\frac{1}{\xi}\right)^N \tag{16}$$

and $\forall n = 1, \cdots, N$

$$\left|\frac{\widetilde{\gamma_n \rho}}{2\pi}\right| < \xi \tag{17}$$

This theorem is a consequence of a Dirichlet pigeonhole argument. Bohr and Landau, (*Göttinger Nachtrichten*, 1910, pages 303-330) are credited with being the first to recognize the relevance of this Dirichlet theorem to the problem of estimating exponential sums involving γ_n.

3.6 Using the Dirichlet Theorem to Analyze F Further

Take

$$\eta = \rho + \xi \quad \text{and} \quad N = \frac{a}{\xi}$$

We consider the quantity

$$\begin{aligned}
\mid -\Im F(\xi + i\eta) + \Im F(\xi + i\xi) \mid \quad &= \left| \sum_{n=1}^{\infty} \frac{e^{-\gamma_n(\xi + i\eta)}}{\gamma_n} - \sum_{n=1}^{\infty} \frac{e^{-\gamma_n(\xi + i\xi)}}{\gamma_n} \right| \\
&= \left| \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \eta - \sum_{n=1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \xi \right| \\
&= \underbrace{\left| \sum_{n=1}^{N} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \eta - \sum_{n=1}^{N} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \xi \right|}_{C} \\
&\quad + \underbrace{\left| \sum_{n=N+1}^{+\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \eta - \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \xi \right|}_{D}
\end{aligned}$$

We shall simplify quantities C and D in turn.

$$\begin{aligned}
C &= \left| \sum_{n=1}^{N} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \eta - \sum_{n=1}^{N} \frac{e^{-\gamma_n \xi}}{\gamma_n} \sin \gamma_n \xi \right| \\
&= \left| \sum_{n=1}^{N} \frac{\sin \gamma_n \eta - \sin \gamma_n \xi}{\gamma_n} \underbrace{e^{-\gamma_n \xi}}_{\leq 1} \right| \\
&\leq \left| \sum_{n=1}^{N} \frac{\sin \gamma_n \eta - \sin \gamma_n \xi}{\gamma_n} \right| \\
&\leq \sum_{n=1}^{N} \frac{|\sin \gamma_n \eta - \sin \gamma_n \xi|}{\gamma_n}
\end{aligned}$$

$$\begin{aligned}
D &= \left| \sum_{n=N+1}^{+\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \underbrace{\sin \gamma_n \eta}_{\leq 1} - \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \underbrace{\sin \gamma_n \xi}_{\leq 1} \right| \\
&= \left| \sum_{n=N+1}^{+\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} - \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \right| \\
&\leq 2 \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n}
\end{aligned}$$

$$\begin{aligned}
| -\Im F(\xi + i\eta) + \Im F(\xi + i\xi) \, | &= C + D \\
&\leq \sum_{n=1}^{N} \frac{|\sin \gamma_n \eta - \sin \gamma_n \xi|}{\gamma_n} \\
&\quad + 2 \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi}}{\gamma_n} \qquad (18)
\end{aligned}$$

By our selection of $\eta = \rho + \xi$,

$$\sin \gamma_n \eta = \sin(\gamma_n \xi + \gamma_n \rho)$$

Since $\gamma_n \rho \geq 0$ and $\sin x \leq x$ when $x \geq 0$ it follows that

$$| \sin \gamma_n \eta - \sin(\gamma_n \xi) | \leq \gamma_n \rho$$

The conclusion of the Dirichlet theorem (17) implies

$$|\gamma_n \rho| \leq 2\pi \xi$$

implying that

$$|\sin\gamma_n\eta - \sin(\gamma_n\xi)| \leq 2\pi\xi$$

Applying this to simplify (18),

$$\begin{aligned}
|-\Im F(\xi+i\eta)+\Im F(\xi+i\xi)\,| &\leq \sum_{n=1}^{N}\frac{\overbrace{|\sin\gamma_n\eta-\sin\gamma_n\xi|}^{\leq 2\pi\xi}}{\gamma_n}+2\sum_{n=N+1}^{\infty}\frac{e^{-\gamma_n\xi}}{\gamma_n}\\
&\leq \underbrace{2\pi\xi\sum_{n=1}^{N}\frac{1}{\gamma_n}}_{O(1)}+2\underbrace{\sum_{n=N+1}^{\infty}\frac{e^{-\gamma_n\xi}}{\gamma_n}}_{\leq\frac{1}{32}\log\left(\frac{1}{\xi}\right)}\\
&\leq O(1)+\frac{1}{16}\log\left(\frac{1}{\xi}\right)
\end{aligned}$$

Since we can control the $O(1)$ constant, it follows that if we take ξ small enough,

$$< \frac{1}{12}\log\left(\frac{1}{\xi}\right)$$

So we conclude

$$|\,-\Im F(\xi+i\eta)+\underbrace{\Im F(\xi+i\xi)}_{\frac{1}{8}\log\frac{1}{\xi}+O(1)}\,| < \frac{1}{12}\log\left(\frac{1}{\xi}\right)$$

which implies that

$$\begin{aligned}
-\Im F(\xi+i\eta) &> \left(\frac{1}{8}-\frac{1}{12}\right)\log\frac{1}{\xi}\\
&= \frac{1}{24}\log\frac{1}{\xi}
\end{aligned}$$

So we have shown that if ξ is small enough i.e. so that $\frac{1}{8}\log\frac{1}{\xi}$ is a good approximation for $F(\xi+i\xi)$, and the hypotheses (16) of the Dirichlet theorem are met, namely that

$$\rho_0 < \eta < \xi + \rho_0\left(\frac{1}{\xi}+1\right)^{\frac{a}{\xi}} \tag{19}$$

Then

$$-\Im F(\xi+i\eta) \;>\; \frac{1}{24}\log\frac{1}{\xi} \tag{20}$$

This numeric lower bound translates into a Ω lower bound, for *suppose the formula*

$$-\Im F(\xi + i\eta) = \Omega_{L|R}(\log\log\eta) \tag{21}$$

is false, to obtain a contradiction. Then

$$\forall K > 0,\ \exists N_K > 0 \text{ such that } \forall \eta > N_K,\ -\Im F(\xi + i\eta) < K \log\log\eta \tag{22}$$

If we take $\rho = N_K$, then all η that satisfy the hypothesis of the Dirichlet theorem (19) will be strictly larger than N_K, so the assertion in (22) may be rewritten as

$$-\Im F(\xi + i\eta) < K \log\log\left\{\xi + \rho_0\left(\frac{1}{\xi} + 1\right)^{\frac{a}{\xi}}\right\}$$

Now we let $\xi \to 0$ and note that

$$\begin{aligned}
\lim_{\xi\to 0}\log\log\left\{\xi + \rho_0\left(\frac{1}{\xi}+1\right)^{\frac{a}{\xi}}\right\} &= \lim_{\mu=1/\xi\to+\infty}\log\log\left\{\underbrace{1/\mu}_{\to 0} + \rho_0\left(\mu+1\right)^{\mu a}\right\} \\
&= \lim_{\mu\to+\infty}\log\left\{\log\rho_0 + \mu a\log\left(\mu+1\right)\right\} \\
&= \lim_{\mu\to+\infty}\log\left\{\underbrace{\log\rho_0}_{O(1)} + \mu a\underbrace{\log\left(\mu+1\right)}_{\to\log\mu}\right\} \\
&= \lim_{\mu\to+\infty}\log\left\{O(1) + \mu\log\mu\right\} \\
&= \lim_{\mu\to+\infty} O(1) + \log\mu \\
&= \lim_{\xi=1/\mu\to 0} O(1) + \log\frac{1}{\xi}
\end{aligned}$$

Which *contradicts* (20), implying the hypothesis that (21) is false is an invalid one, i.e. we have shown that

$$-\Im F(\xi + i\eta) = \Omega_{L|R}(\log\log\eta)$$

3.7 A Modification of the Phragmén-Lindelöf Theorem

Recall that our ultimate task (13) involves showing that

$$-\Im F(i\eta) = \Omega_{R|L}(\log\log\eta) \tag{23}$$

whereas so far we have only succeeded in showing that for all $\xi > 0$,

$$-\Im F(\xi + i\eta) = \Omega_{R|L}(\log\log\eta) \tag{24}$$

Now if $F(z)$ was regular for $\xi \geq 0$, or if $F(z)$ was regular for $\xi > 0$ and continuous for $\xi \geq 0$ the situation would satisfy the hypotheses of the Phragmén-Lindelöf Theorem and we would be able to immediately deduce (23) from (24).

In the present situation, unfortunately, $F(z)$ is *not continuous* for $\xi \geq 0$.

To rectify this situation Littlewood proves the following modified form of the Phragmén-Lindelöf Theorem, whose hypotheses are satisfied by $F(z)$, and whose conclusion is strong enough to justify the desired deduction.

Theorem. (Littlewood). Suppose the following 5 conditions hold:

1. $f(z)$ is regular in the open semi-infinite strip $z = (\xi + i\eta)$

$$0 < \xi < 1 \ , \ \eta \geq \eta_0 > 0 \tag{25}$$

2. $f(\xi + i\eta)$ tends to a limit as $\xi \to 0$, for each value of η, and that positive constants A_1, A_2, and p exist such that
3. given any number $y > \eta_0$ we can find a positive number δ_y such that

$$\left| \frac{f(\xi + i\eta)}{f(i\eta)} \right| < A_1$$

   for all

$$0 \leq \xi \leq \delta_y \ , \ \eta_0 \leq \eta \leq y$$

4. on the boundary of the strip

$$|f(z)| < A_2$$

5. in the interior of the strip

$$|f(z)| = O(e^{z^p})$$

then there is a constant A such that

$$f(z) < A$$

in the interior *and* on the boundary of the strip.

Proof.

The quantities in the argument that follows may be "scaled" appropriately. We shall take η_0 sufficiently large.

Let us choose a q satisfying

$q > p$, and

$$q \text{ arctan } \left(\frac{1}{\eta_0}\right) < \frac{1}{2}\pi$$

Now if $z = re^{i\theta}$ is in the strip $0 < r\sin\theta < 1$, $\eta \geq r\cos\theta > 0$, then

$$\frac{1}{2}\pi - \text{ arctan } \left(\frac{1}{\eta_0}\right) \leq \theta \leq \frac{1}{2}\pi$$

which implies

$$\cos q\ (\theta - \frac{1}{2}\pi) > 0$$

and

$$\Re\ (-iz)^q > 0$$

For any $\epsilon > 0$, define

$$\Phi(z) = f(z)e^{-\epsilon(-iz)^q}$$

Then on all points of the boundary of the strip

$$|\Phi(z)| \leq |f(z)| < A_2 \tag{26}$$

and furthermore as $\eta \to \infty$, $\Phi(z) \to 0$ uniformly for all $0 \leq \xi \leq 1$. We can therefore choose a value for y such that

$$\forall\xi \in [0,1], \quad |\Phi(\xi + iy)| < A_2$$

and this value of y can be as large as we please. From this we can conclude that (26) holds for all z on the boundary of the rectangle R whose corners are $(0,\eta_0)$, $(1,\eta_0)$, $(1,y)$ and $(0,y)$.

Condition (3) of the hypotheses provides us with a certain δ_y. We divide the rectangle R into

$$\begin{aligned} R' &= \{z \in R \mid \Re\ z \leq \delta_y\}, \text{ and} \\ R'' &= \{z \in R \mid \Re\ z \geq \delta_y\} \end{aligned}$$

Condition (3) of the hypotheses guarantees that δ_y is chosen such that

$$\forall z \in R', \quad \Phi(z) \leq A_1 A_2 \tag{27}$$

and, in this case, we can verify that

$$A_1 \geq 1$$

Thus, (27) holds also on the boundary of R'', and since, by assumption, $\Phi(z)$ is regular in and on the boundary of R'', it follows that (27) holds inside of R'' as well.

Having showed that (27) holds in all of rectangle R, we now let $\epsilon \to 0$ (as in the proof of the classical Phragmén-Lindelöf Theorem, and conclude that

$$|f(z)| < A = A_1 A_2$$

both inside and on the boundary of R. This proves the theorem, with $A = A_1 A_2$.
□

3.8 F near 0, off the line $\Re(z) = \Im(z)$

With the modified Phragmén-Lindelöf Theorem, Littlewood considers

$$-\Im F(i\eta) = \lim_{\xi \to 0} \ -\Im F(\xi + i\eta)$$

The proof is by contradiction. Suppose that

$$-\Im F(i\eta) = \Omega_R(\log\log\eta) \tag{28}$$

is false. Then for any positive number L, there exists a positive number N_L such that

$$\forall \eta > N_L, \ \ -\Im F(i\eta) \ \ < \ \ L\log\log\eta \tag{29}$$

Let

$$f(z) \ \stackrel{\text{def}}{=} \ e^{iF(z)}(\log z)^{-L'} \ \ , \text{ for any } L' > L$$

We show that $f(z)$ above satisfies hypothesis 1-5 of the modified Phragmén-Lindelöf Theorem.

1. **Lemma.** $f(\xi + i\eta)$ is regular in the open semi-infinite strip $z = (\xi + i\eta)$, $0 < \xi < 1$, $\eta \geq \eta_0 > 0$.
   Proof. Follows immediately from the definition of $f(\xi + i\eta)$. □

2. **Lemma.** $f(\xi + i\eta)$ tends to a limit as $\xi \to 0$, for each value of η.
   Proof. Follows immediately from the definition of $f(\xi + i\eta)$. □

3. **Lemma.** positive constants A_1, A_2, and p exist such that given any number $y > \eta_0$ we can find a positive number δ_y such that

   $$\left|\frac{f(\xi + i\eta)}{f(i\eta)}\right| < A_1$$

   for all

   $$0 \leq \xi \leq \delta_y \ \ , \ \ \eta_0 \leq \eta \leq y$$

   Proof. We know that uniformly for all $T > x^2 = e^{2\eta}$,

   $$\sum_{\gamma_n > T} \frac{\sin\gamma_n\eta}{\gamma_n} = O(1)$$

   Let us choose N such that $\gamma_{N+1} > e^{2y}$. Then uniformly for all $\nu > N$, $2 \leq \eta \leq y$

   $$\sum_{n=\nu}^{\infty} \frac{\sin\gamma_n\eta}{\gamma_n} = O(1)$$

Now by a partial summation, it follows that uniformly for all $\xi \geq 0$ and $2 \leq \eta \leq y$,

$$\sum_{n=N+1}^{\infty} e^{-\gamma_n \xi} \frac{\sin \gamma_n \eta}{\gamma_n} = O(1)$$

Thus

$$\begin{aligned} |-\Im F(\xi + i\eta) + \Im F(i\eta)| &= \left| \sum_{n=1}^{\infty} (1 - e^{-\gamma_n \xi}) \frac{\sin \gamma_n \eta}{\gamma_n} \right| \\ &\leq N\xi + \left| \sum_{n=N+1}^{\infty} \frac{\sin \gamma_n \eta}{\gamma_n} \right| + \left| \sum_{n=N+1}^{\infty} \frac{e^{-\gamma_n \xi} \sin \gamma_n \eta}{\gamma_n} \right| \\ &= N\xi + O(1) \end{aligned}$$

and

$$\begin{aligned} \left| \frac{f(\xi + i\eta)}{f(i\xi)} \right| &= e^{-\Im F(\xi + i\eta) + \Im F(i\eta)} \left| \frac{\log(i\eta)}{\log(\xi + i\eta)} \right|^K \\ &< K_1 e^{N\xi + K_2} \end{aligned}$$

where K_1 and K_2 are constants. So condition (iii) is satisfied by taking $\delta_y = 1/N$. □

4. **Lemma.** on the boundary of the strip

$$|f(z)| < A_2$$

Proof. Follows from (29). □

5. **Lemma.** in the interior of the strip

$$|f(z)| = O(e^{z^p})$$

Proof. It is known that

$$\frac{\psi(x) - x}{\sqrt{x}} = O\,(\log x)^2$$

therefore, uniformly for all ν such that $\gamma_\nu > x^2 = e^{2\eta}$

$$\sum_{n=1}^{\nu} \frac{\sin \gamma_n \eta}{\gamma_n} = O(\eta^2) \tag{30}$$

On the other hand if $\gamma_\nu \leq e^{2\eta}$

$$\begin{aligned} \sum_{n=1}^{n=\nu} \frac{\sin \gamma_n \eta}{\gamma_n} &= O \sum_{\gamma_n < e^{2\eta}} \frac{1}{\gamma_n} \\ &= O \sum_{k < e^{2\eta}} \frac{\log k}{k} \\ &= O(\eta^2) \end{aligned}$$

This shows that (30) holds uniformly, for all values of ν. Then, by a partial summation we conclude that

$$-\Im F(\xi+i\eta)=\sum_{1}^{\infty}\frac{e^{-\gamma_n\xi}\sin(\gamma_n\eta)}{\gamma_n}=O(\eta^2)$$

implying

$$f(z)=e^{O(\eta^2)}\cdot o(1)=O(\eta^2)$$

Thus condition by taking $p=3$, the lemma is proved. □

Since all 5 hypothesis are met, the conclusion of the modified Phragmén-Lindelöf Theorem holds regarding $f(z)$. In particular then, for $z=\xi+i\eta$, with $0\le\xi\le 1$ and $\eta\ge 2$,

$$f(z)=O(1)$$

implying that

$$e^{iF(z)}=f(z)(\log z)^{L'}=O\left\{(\log z)^{L'}\right\}$$

from which it follows that

$$-\Im F(z)\le 2L'\log\log\eta \tag{31}$$

for all sufficiently large values of η. But L is an arbitrary positive number and so $L'>L$ can also be chosen arbitrarily small. So if we take $L=1/64$, $L'=1/32$ then (31) *contradicts* a previously proven assertion (15) that

$$-\Im F(\xi+i\xi)=\frac{1}{8}\log\frac{1}{\xi}+O(1)$$

Thus, the assumption that (28) is false must be incorrect. In fact then we have shown

$$-\Im F(i\eta)=\Omega_R(\log\log\eta)$$

3.9 Concluding Steps of the Proof

We have shown

$$\begin{aligned}-\Im F(i\eta) &= \sum_{n=1}^{\infty}\frac{\sin(\gamma_n\eta)}{\gamma_n}\\ &= \Omega_R(\log\log\eta)\end{aligned}$$

implies

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin(\gamma_n\log x)}{\gamma_n} &= -\Im F(i\log x)\\ &= \Omega_R(\log\log\log x)\end{aligned}$$

Our starting point, from Landau,

$$\frac{\psi(x) - x}{\sqrt{x}} = -2\sum_{n=1}^{\infty} \frac{\sin \gamma_n \log x}{\gamma_n} + O(1)$$

And thus the Ω bounds on $-\Im F(i\eta)$ give rise to corresponding Ω bounds on $\psi(x) - x$,

$$\begin{aligned} \psi(x) - x &= \Omega_R(\sqrt{x} \log\log\log x) \\ \psi(x) - x &= \Omega_L(\sqrt{x} \log\log\log x) \end{aligned}$$

4 Final Remarks and Subsequent Progress

Since Littlewood's original proofs of these theorems, several more sophisticated "simplifications" of the argument have been presented, due in chronological order to Ingham, Diamond and Halász.

One consequence of Littlewood's theorems is that they show that the sign of $\pi(x) - x$ changes infinitely often, (of course the theorems give much more specific information that just this). While we have not shown it here, this fact carries over to the identical conclusion regarding the sign changes of $\pi(x) - x$. In showing this, Littlewood disproved a conjecture of Riemann, that $\forall x, \pi(x) \leq \mathrm{Li}(x)$.

Unfortunately, Littlewood's theorem is an "existence theorem", which is to say that to date, there has been no success at finding a natural number x_0 for which $\pi(x_0) > \mathrm{Li}(x_0)$. The harder problem of locating the first sign change of $\pi(x) - \mathrm{Li}(x)$ remains impregnable. It was shown by Skewes in 1933 that there is an $x_0 < 10_4(3)$ for which $\pi(x_0) > \mathrm{Li}(x_0)$ holds[2]. While an explicit x_0 is still not known, the interval in which we are assured that such an x_0 exists has shrunk since Skewes original work, notably due to the work of Lehman (1966) and Riele (1986).

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[2] $10_1(x) = 10^x$, $10_2(x) = 10^{10_1(x)}$, $\cdots$. A sketch of Skewes argument is presented in Littlewood's *Miscellany.*

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