Positively generated subgroups of free groups and the Hanna Neumann conjecture

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Abstract. The *Hanna Neumann conjecture* states that if F is a free group, then for all subgroups $H, K \leqslant F$,

$$\mathrm{rank}(H \cap K) - 1 \leqslant [\mathrm{rank}(H) - 1]\,[\mathrm{rank}(K) - 1]$$

Previous research on the conjecture has proceeded largely by "translating" the group-theoretic properties of subgroups of free groups into the graph-theoretic properties of their corresponding foldings or finite automata. This paper attempts to elaborate the reverse.

In particular, in this paper we give group-theoretic interpretation of the well-known graph-theoretic property of strong connectivity. Specifically, we show that strong connectivity of a subgroup's folding corresponds exactly to the property that the subgroup is *positively generated* (i.e. is generated by a set of words containing no negative exponents). To accomplish this, we present the notion of a strong directed trail decomposition of a directed graph; this decomposition provides a useful computational tool, and facilitates inductive arguments about the properties of positively generated subgroups of free groups.

As an example application of directed trail decomposition techniques, we prove that if a subgroup $H \leqslant F$ is positively generated, or if its associated folding Γ_H has no source or sink vertices, then for all subgroups $K \leqslant F$, the Hanna Neumann conjecture holds for the pair (H, K). We also show that if a subgroup of a free group is positively generated, then it has a positive basis. Finally, we describe an algorithm which decides whether an arbitrary finitely generated subgroup of a free group is positively generated, and if so, outputs a positive basis for the subgroup.

1. Introduction

Improving Howson's earlier bound [**5**] on the rank of intersections of finitely generated (f.g.) subgroups of free groups, H. Neumann proved in [**9**] that any $H, K \leqslant_{\text{f.g.}} F$ must satisfy

$$rank(H \cap K) - 1 \leqslant 2[rank(H) - 1][rank(K) - 1]$$

The stronger assertion obtained by omitting the factor of 2 has come to be known as the Hanna Neumann conjecture. In [**1**], Burns improved H. Neumann's bound

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by showing that in fact

$$\begin{aligned} rank(H \cap K) - 1 \quad &\leqslant \quad 2[rank(H) - 1][rank(K) - 1] \\ &\quad\quad -min(rank(H) - 1, rank(K) - 1) \end{aligned}$$

In 1983, J. Stallings introduced the notion of a *folding* and showed how to apply these objects to the study of subgroups of free groups [**15**]: Recall the well-defined constructive map which assigns to each finitely generated subgroup H of a free group $F = F(X)$, a corresponding folding $\Gamma_H = (V_H, E_H)$. We view the folding Γ_H as a *deterministic finite automaton*, represented as a directed multigraph (with loops), with each directed edge labeled by an element of the ground set X. The folding Γ_H enjoys the property that the set of freely reduced elements in H coincides with the set of words that can be read along closed non-backtracking walks that start and end at a distinguished vertex $1_H \in V_H$.

Stallings's approach was applied by Gersten in [**4**] to solve certain special cases of the conjecture. Similar techniques were developed over a sequence of papers by Imrich [**7, 6**], Nickolas [**11**], and Servatius [**13**] who gave alternate proofs of Burns' bound and resolved special cases of the conjecture. In 1989, W. Neumann showed that the conjecture is, in a sense, true "with probability 1" for randomly chosen subgroups of free groups [**10**], and proposed a stronger form of the conjecture. In 1992, Tardos proved in [**16**] that the conjecture is true if one of the two subgroups has rank 2. In 1994, Warren Dicks showed that the strong Hanna Neumann conjecture is equivalent to a conjecture on bipartite graphs, which he termed the Amalgamated Graph conjecture [**2**]. In 1996, Tardos used Dicks' method to give the first new bound for the general case in [**17**], where he proved that $\forall H, K \leqslant_{\text{f.g.}} F$,

$$\begin{aligned} rank(H \cap K) - 1 \quad &\leqslant \quad 2[rank(H) - 1][rank(K) - 1] \\ &\quad\quad -rank(H) - rank(K) + 1 \end{aligned}$$

This is the best known bound for the general case; the conjecture remains open.

To date, research on the Hanna Neumann conjecture has focused largely on "translating" the group-theoretic properties of subgroups of free groups into the graph-theoretic properties of their corresponding foldings. In 1999, at the NY Group Theory Seminar, A. Miasnikov proposed a research project to elaborate the reverse, i.e. to interpret well-known properties of graphs in group-theoretic terms. In this paper, we provide a group-theoretic interpretation of the well-known graph-theoretic property of *strong connectivity*:

DEFINITION 1.1. Given a directed graph $\Gamma = (V, E)$ define the binary equivalence relation $SC \subseteq V \times V$ (**strong connectivity**). Specifically, for $u, v \in V$

$$(u, v) \in SC \leftrightarrow \begin{array}{l} u = v, \text{ or} \\ \text{[There is a directed path from } u \text{ to } v, \\ \text{and there is a directed path from } v \text{ to } u] \end{array}$$

A directed graph $\Gamma = (V, E)$ is **strongly connected** iff $SC = V \times V$.

DEFINITION 1.2. Given a word $w \in F(X) = F(\{x_1, \ldots, x_n\})$, we say that w is **positive** (with respect to basis X) if its freely reduced form consists only of the symbols $x_1, \ldots, x_n$ (and contains no occurrence of $x_1^{-1}, \ldots, x_n^{-1}$).

DEFINITION 1.3. A subgroup $H \leqslant F(X)$ is said to have a positive generating set (or simply: "H is **positively generated**") if $\exists S \subseteq H$ such that $\langle S\rangle = H$ and $\forall w \in S$, w is positive. Note that unless explicitly stated, S need not be a basis.

We shall demonstrate that *strong connectivity* of the folding of a subgroup corresponds precisely to the property that the subgroup is *positively generated.* To do this, we shall first define the directed trail decomposition of a directed graph, and demonstrate that strong connectivity of a folding is equivalent to it possessing a certain decomposition of this type. Foldings of positively generated subgroups of free groups will then be shown to be strongly connected, and hence to possess such decompositions. The decompositions provide a useful computational tool, and enable inductive arguments about the properties of positively generated subgroups of free groups.

As an example application of directed trail decomposition techniques, we will prove that if a subgroup H is positively generated, or if its associated folding Γ_H has no source or sink vertices, then for all subgroups K, the Hanna Neumann conjecture holds for the pair H, K. This result was obtained independently using different techniques by Prof. J. Meakin, and was described by him at the Albany Group Theory Conference 2000, where this work was also presented.

2. Results

2.1. Strong connectivity & directed trail decompositions. In this section we define (strong) directed trail decompositions and show that they are intimately related to strong connectivity of directed graphs. These decompositions will be used in later sections to prove statements about positively generated subgroups of free groups.

DEFINITION 2.1. A **directed trail** P in a directed graph $\Gamma = (V, E)$ is a non-empty sequence of distinct directed edges $e_1, e_2, \ldots, e_m$ ($e_i \in E$, $i = 1, \ldots, m$) for which $tail(e_{i+1}) = head(e_i)$ ($i = 1, \ldots, m-1$). The length of P is denoted $|P| = m$. The *start* of P is denoted as $s(P) = tail(e_1)$, and the *terminus* of P as $t(P) = head(e_{|P|})$.

DEFINITION 2.2. A **self-avoiding directed trail** $P = (e_1, e_2, \ldots, e_m)$ in a directed graph $\Gamma = (V, E)$ is a directed trail which additionally satisfies for all $i, j \in \{1, \ldots, m\}$, $i \neq j \Rightarrow [tail(e_i) \neq tail(e_j)$ and $head(e_i) \neq head(e_j)]$. Notice that by this definition, a self-avoiding trail may satisfy $head(e_m) = tail(e_1)$.

REMARK 2.3. It is easy to verify that given a directed trail $P = (e_1, e_2, \ldots, e_{|P|})$ in $\Gamma = (V, E)$, there is always a *self-avoiding* trail $P' = (f_1, f_2, \ldots, f_{|P'|})$ where for $i = 1, \ldots, |P|$, $f_i \in \{e_1, e_2, \ldots, e_{|P|}\}$ and $s(P) = s(P')$, $t(P) = t(P')$.

DEFINITION 2.4. A sequence of directed trails $\mathbf{P} = (P_0, \ldots, P_n)$ in a directed graph $\Gamma = (V, E)$ is called a **directed trail decomposition** of Γ if it satisfies the following 3 conditions:

(1) The trails are a partition of the edges of Γ:
$$\bigcup_{i=0}^{n} P_i = E \text{ and } i \neq j \Rightarrow P_i \cap P_j = \emptyset$$
(2) $s(P_0) = t(P_0)$, and we denote this vertex as 1_Γ.

(3) For each $i = 1, \ldots, n$, the directed trail P_i satisfies:

$$V[P_i] \cap \bigcup_{j=0}^{i-1} V[P_j] \neq \emptyset \quad \Rightarrow \quad V[P_i] \cap \bigcup_{j=0}^{i-1} V[P_j] = \{s(P_i), t(P_i)\}$$

$$V[P_i] \cap \bigcup_{j=0}^{i-1} V[P_j] = \emptyset \quad \Rightarrow \quad s(P_i) = t(P_i)$$

The above decomposition naturally extends Whitney's definition of "ear decompositions" of 2-edge connected *undirected* graphs [**19**] to the class of *directed* graphs.

DEFINITION 2.5. A **strong directed trail decomposition** of a directed graph $\Gamma = (V, E)$ is a directed trail decomposition $P_0, \ldots, P_n$ of Γ which satisfies

$$\forall i \in \{1, \ldots, n\}, \;\; V[P_i] \cap \bigcup_{j=0}^{i-1} V[P_j] \neq \emptyset$$

Strong directed trail decompositions are the natural *directed* counterpart of Whitney's "open ear decomposition" for 2-vertex connected *undirected* graphs [**19**].

The next two lemmas demonstrate that strong connectivity of a directed graph Γ is equivalent to the existence of a strong directed trail decomposition of Γ.

LEMMA 2.6. *If $\Gamma = (V, E)$ has a strong directed trail decomposition, then Γ is a strongly connected directed graph.*

PROOF. Let $P_0, \ldots, P_n$ be a strong directed trail decomposition of Γ. Let $v_0 \in V$ be arbitrary; we show there is a directed trail from v_0 to 1_Γ and a directed trail from 1_Γ to v_0.

Clearly, $v_0 \in P_{i_0}$ for some $i_0 \in \{0, \ldots, n\}$. Successively, for each $m \geqslant 1$, if $i_{m-1} \neq 0$ we define $v_m = t(P_{i_{m-1}})$ and choose $i_m < i_{m-1}$ such that $v_m \in P_{i_m}$. Since $i_0, i_1, \ldots, i_m, \ldots$ are monotonically decreasing indices from the finite set $\{0, \ldots, n\}$, there is some M sufficiently large for which $i_{M-1} = 0$, and hence $v_M = 1_\Gamma$. By concatenating final segments of the directed trails $P_{i_0}, P_{i_1}, \ldots, P_{i_{M-1}}$, we obtain the directed trail

$$v_0 \overset{P_{i_0}}{\rightsquigarrow} v_1 \overset{P_{i_1}}{\rightsquigarrow} \ldots \overset{P_{i_{M-1}}}{\rightsquigarrow} v_M = 1_\Gamma$$

which connects v_0 to 1_Γ.

Put $j_0 = i_0$. Successively, for each $\ell \geqslant 1$, if $j_{\ell-1} \neq 0$ we define $u_\ell = s(P_{j_{\ell-1}})$ and choose $j_\ell < j_{\ell-1}$ such that $u_\ell \in P_{j_\ell}$. Since $j_0, j_1, \ldots, j_\ell, \ldots$ are monotonically decreasing indices from the finite set $\{0, \ldots, n\}$, there is some L sufficiently large for which $j_{L-1} = 0$, and hence $u_L = 1_\Gamma$. By concatenating initial segments of the directed trails $P_{j_{L-1}}, P_{j_{L-2}}, \ldots, P_{j_0}$, we obtain the directed trail

$$1_\Gamma = u_L \overset{P_{j_{L-1}}}{\rightsquigarrow} u_{L-1} \overset{P_{j_{L-2}}}{\rightsquigarrow} \ldots \overset{P_{j_1}}{\rightsquigarrow} u_1 \overset{P_{j_0}}{\rightsquigarrow} v_0$$

connecting 1_Γ to v_0.

□

The converse of lemma 2.6 is also true, as we now show.

LEMMA 2.7. *If $\Gamma = (V, E)$ is a strongly connected directed graph, then Γ has a strong directed trail decomposition consisting of self-avoiding directed trails.*

PROOF. Fix an arbitrary vertex in Γ, and denote it as 1_Γ. We give the following effective procedure for constructing a directed trail decomposition. First, define $\Gamma_0 \overset{\text{def}}{=} (1_\Gamma, \emptyset)$. Then, starting with $i = 0$:

(1) If $V[\Gamma_i] = V$, proceed to step 2. Otherwise, fix any $v \in V \backslash V[\Gamma_i]$. Since Γ is strongly connected, we can choose a directed trail s from 1_Γ to v, and a directed trail t from v to 1_Γ. Suppose that s is the sequence of edges $(s_1, s_2, \dots, s_{|s|})$, where $tail(s_1) = 1_\Gamma$ and $head(s_{|s|}) = v$. Fix s_j to be the last edge in s for which $tail(s_j) \in V[\Gamma_i]$; put $x_i = tail(s_j)$. Similarly, suppose that t is the sequence of edges $(t_1, \dots, t_{|t|})$, where $tail(t_1) = v$ and $head(t_{|t|}) = 1_\Gamma$. Fix t_k to be the first edge in t for which $head(t_k) \in V[\Gamma_i]$; put $y_i = head(t_k)$. Define

$$P_i = (s_j, s_{j+1}, \dots, s_{|s|}, t_1, \dots t_{k-1}, t_k)$$

Clearly P_i is a trail from x_i to y_i. In light of remark 2.3, we may (by suitably adjusting our choice of v, s and t) assume that P_i is a self-avoiding trail. Put $\Gamma_{i+1} = \Gamma_i \sqcup P_i$. Increment i. Repeat step 1.

(2) If $E[\Gamma_i] = E$, halt. Otherwise, fix any $e_i = (x_i, y_i) \in E \backslash E[\Gamma_i]$. Take the ith directed trail to be $P_i = (e_i)$ and put $\Gamma_{i+1} = \Gamma_i \sqcup P_i$. Increment i. Repeat step 2.

Notice that at each iteration of the procedure, the trail P_i is constructed so that it does not contain any edges already in Γ_i. Indeed, for $i \geqslant 0$, P_i attaches to Γ_i at precisely its start and terminus vertices x_i, y_i respectively. Thus, the procedure outputs a strong directed trail decomposition of Γ that consists of self-avoiding directed trails.

□

We shall later need the following technical refinement of the above lemma.

COROLLARY 2.8. If $\Gamma = (V, E)$ is a strongly connected directed graph, then Γ has a strong directed trail decomposition consisting of self-avoiding directed trails $P_0, \dots, P_n$ such that $\Gamma_0 = (1_\Gamma, \emptyset)$ and $\Gamma_i = \bigcup_{j=0}^{i-1} P_j$ (for $i = 1, \dots n+1$) are strongly connected directed graphs.

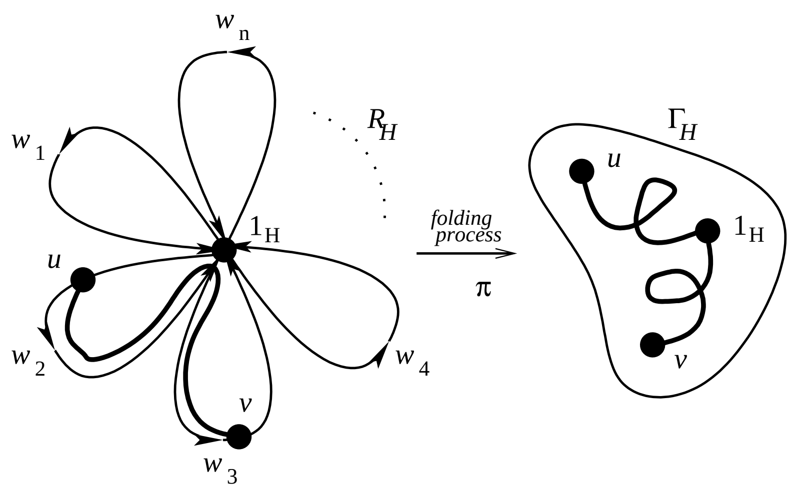
PROOF. Note that in the proof of lemma 2.7, for each $i = 1, \dots n+1$, the sequence $P_0, \dots, P_{i-1}$ forms a strong directed trail decomposition of Γ_i. Thus, by lemma 2.6, each Γ_i $(i = 0, \dots, n+1)$ is a strongly connected directed graph. □

2.2. Strong connectivity and positive generating sets. In this section we describe the close connection between the class of positively generated finite rank subgroups of a free group, and the class of strongly connected foldings. To begin, the next lemma shows that if a finitely generated subgroup H of a free group F is generated by a set of positive words, then H must necessarily have a strongly connected folding.

LEMMA 2.9. *Let H be a finitely generated subgroup of a free group $F(X)$, and let $\Gamma = (V, E)$ be the folding of H. If H is positively generated (with respect to basis X), then Γ_H is a strongly connected directed graph.*

Proof. Let $S = \{w_1, \cdots, w_n\}$ be a positive generating set for H, i.e. $H = \langle S \rangle$, where w_i is positive for all $i = 1, \ldots, n$.

Let R_H be the graph obtained as follows: (1) Construct n directed cycles $c_1 = (V_1, E_1)$, ..., $c_n = (V_n, E_n)$, where $|V_i| = |w_i|$. (2) Pick one vertex from each of the cycles, and identify this subset of vertices; denote the resulting vertex 1_H. (3) Label cycle c_i's edges by successive letters of w_i, starting at vertex 1_H. We call the resulting labelled directed graph R_H the *rose* of H. Because the generating set S consists of positive words, R_H contains a directed path $p_{u,v}$ between any two vertices u and v—simply take $p_{u,v}$ to be the path that goes from u to the vertex 1_H followed by the path from 1_H to v.



Now recall the "folding process" π that transforms R_H into the folding Γ_H: Repeatedly identify pairs of edges e, e' which satisfy

$$label(e) = label(e') \ \wedge \ [head(e) = head(e') \vee tail(e) = tail(e')]$$

It is easy to see that if w is any freely reduced word that can be read in R_H along a trail that starts at vertex u and ends at vertex v, then w can also be read in Γ_H along a trail that starts at vertex $\pi(u)$ and ends at vertex $\pi(v)$. Since positive words are necessarily freely reduced, it follows that Γ_H is strongly connected.

□

The converse of lemma 2.9 is also true. In fact, we prove a statement that is (a priori) stronger.

Lemma 2.10. *Let H be a finitely generated subgroup of a free group F, and let $\Gamma = (V, E)$ be the folding of H. If Γ_H is a strongly connected directed graph, then H has a basis consisting of positive words.*

Proof. Since Γ_H is a strongly connected directed graph, we know by corollary 2.8, that Γ has a directed trail decomposition $P_0, P_1, \ldots, P_n$ such that $\Gamma_0 = (1_\Gamma, \emptyset)$ and $\Gamma_i = \bigcup_{j=0}^{i-1} P_j$ (for $i = 1, \ldots n+1$) are strongly connected directed graphs.

For $i = 0, \ldots, n$ put $x_i = s(P_i)$ and $y_i = t(P_i)$. Since P_i is part of a *strong* directed trail decomposition, $x_i, y_i \in V_i$. Since Γ_i is strongly connected, fix s_i to be a directed trail in Γ_i from 1_H to x_i, and fix t_i to be a directed trail in Γ_i from y_i to

1_H. Define $w_{P_i}, w_{s_i}, w_{t_i}$ to be the words read along P_i, s_i, and t_i respectively. For $i = 1, \ldots, n$, put $h_i = w_{s_i} \circ w_{P_i} \circ w_{t_i}$. Clearly each h_i is a positive word. Define $B_H = \{h_0, \ldots, h_n\}$.

We show that B_H is a free basis for H: For each $i = 0, \ldots, n$, fix an arbitrary edge c_i in P_i, and define $L_i \subset E_i$ to be the set of edges in P_i excluding c_i. Put $T_0 = (1_\Gamma, \emptyset)$ and $T_i = (V_i, \bigcup_{j=0}^{i-1} L_i)$, for $i = 1, \ldots n+1$. Now T_i is a well-defined subgraph of Γ_i, and T_0 is spanning tree of Γ_0. We assume inductively that T_i is a spanning tree of Γ_i. Then, since P_i is a directed trail which attaches to Γ_i at precisely $s(P_i)$, $t(P_i)$, the omission of edge c_i from L_i suffices to ensure that $T_{i+1} = T_i \cup L_i$ is a spanning tree of $\Gamma_{i+1} = \Gamma_i \cup P_i$. By induction, T_{n+1} is a spanning tree of $\Gamma_{n+1} = \Gamma_H$.

Since B_H consists precisely of the Schreier transversals of H relative to the spanning tree T_{n+1} (where h_i is the transversal of the non-tree edge c_i) it follows that B_H is a free basis for H. $\square$

Combining the results of lemmas 2.9 and 2.10, we see that the existence of a positive generating set for a subgroup $H \leqslant_{\text{f.g.}} F$ is equivalent to the existence of a positive basis for H.

Corollary 2.11. Let H be a finitely generated subgroup of a free group F. H is positively generated iff H has a basis consisting of positive words.

Proof. $\Leftarrow$ Trivial.
$\Rightarrow$ If H is generated by a set of positive words, then by lemma 2.9, Γ_H is strongly connected. Then, by lemma 2.10, H has a basis consisting of positive words. $\square$

Using lemma 2.10 one can show that the class of positively generated subgroups of F extends the class of finite index subgroups of F.

Corollary 2.12. Let H be a finitely generated finite index subgroup of a free group $F = F(X)$. Then H is positively generated.

Proof. Suppose, towards contradition, that H is not positively generated. Then by lemma 2.10, the folding $\Gamma_H = (V_H, E_H)$ is not strongly connected. Let $J_1, \ldots J_m$ denote the equivalence classes of V under the strong connectivity relation SC. Define the ith strongly connected component $\widetilde{J}_i$ to be the subgraph induced by J_i $(i = 1, \ldots, m)$. Let Γ_H^* be the directed labelled multigraph obtained by collapsing each component $\widetilde{J}_i$ to a single vertex u_i. Since Γ_H^* is necessarily a directed acyclic graph, take u_{i_0} to be any minimal vertex (i.e. u_{i_0} has only outgoing edges). Let (s,t) be any edge such that $s \in J_{i_0}$ and $t \notin J_{i_0}$. Suppose (s,t) is labelled by $c \in X$. Since, by assumption, H is finite index, the edge (s,t) must be part of a c-monochromatic directed cycle C in Γ_H. It follows that there is a c-monochromatic directed path $P = C \backslash (s,t)$ from t to s. But since $t \notin J_{i_0}$ and $s \in J_{i_0}$, it must be that there is some edge (t', s') in P where $t' \notin J_{i_0}$ and $s' \in J_{i_0}$. This contradicts our choice of u_{i_0} as a minimal vertex in Γ_H^*. Thus, H is positively generated. $\square$

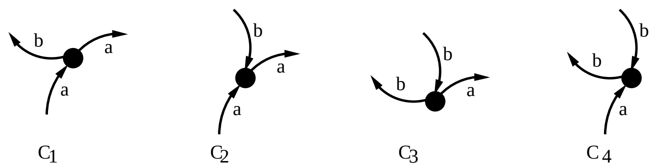
Remark 2.13. Note that unlike finite index subgroups, the class of positively generated subgroups is not closed under intersections. For example,

$$\langle aa, ba \rangle \cap \langle ab, bb \rangle = \langle ab^{-1} \rangle$$

3. Applications to the Hanna Neumann conjecture

In this section, we shall prove that if a subgroup $H \leqslant F$ is positively generated, then for all subgroups K, the Hanna Neumann conjecture holds for the pair (H, K). Directed trail decompositions of foldings will play a central role in the proof.

For concreteness, much of this exposition is restricted to the finitely generated subgroups of $F_2 = F(\{a, b\})$. Suppose we are given H, a non-trivial finitely generated subgroup of F_2, and its folding $\Gamma_H = (V_H, E_H)$. (The reader who wishes to review a standard constructive definition of Γ_H may consult the proof of lemma 2.9 on page 5, where it was outlined.) Since $H \leqslant_{\text{f.g.}} F_2$, Γ_H has vertices of undirected degree $\leqslant 4$ (the undirected degree of a vertex is the sum of its in-degree and out-degree). Define $d = d_H : V_H \to \{1, 2, 3, 4\}$ to be the function that assigns to each vertex $v \in V_H$ its undirected degree in Γ_H. Now put $d_i(\Gamma_H) = |\{v \in V_H |\ \mathrm{d}_H(v) = i\}|$, for $i = 1, 2, 3, 4$. We classify vertices of degree 3 based on the labels of their incident edges, naming the 4 classes C_1, C_2, C_3, and C_4; these classes are shown in the figure below. We define $C_i(\Gamma_H)$ to be the number of degree 3 vertices of type C_i in Γ_H.



Definition 3.1. A folding Γ is called **3-balanced** if it satisfies the following "flow conservation law":

$$C_1(\Gamma) + C_3(\Gamma) = C_2(\Gamma) + C_4(\Gamma) \tag{3.1}$$

We know by lemma 2.9 that a positive finitely generated subgroup of a free group must have a strongly connected folding. By lemma 2.7 we see that strongly connected foldings have strong directed trail decompositions. Now we show that whenever a folding of a subgroup $H \leqslant_{\text{f.g.}} F_2$ has a (not necessarily strong) directed trail decomposition, then this folding is necessarily 3-balanced.

Lemma 3.2. *If $H \leqslant_{f.g.} F_2$ such that $\Gamma_H = (V, E)$ has a directed trail decomposition, then Γ_H is 3-balanced.*

Proof. Suppose Γ_H has a directed trail decomposition $P_0, \ldots, P_n$. Define as before $\Gamma_0 = (1_\Gamma, \emptyset)$ and $\Gamma_i = (V_i, E_i) = \bigcup_{j=0}^{i-1} P_j$ (for $i = 1, \ldots n+1$). Clearly, Γ_{i-1} and P_{i-1} are subgraphs of Γ_i, and $\Gamma_{n+1} = \Gamma_H$. Since (P_j), $j = 0, \ldots, i-1$ is a directed trail decomposition of Γ_i, every vertex in V_i $(i \geqslant 1)$ has degree at least 2.

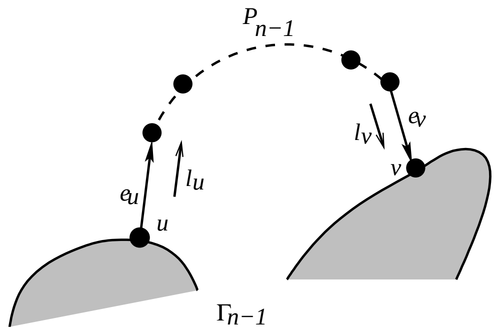
We prove the lemma by induction on n.

Base case: When $n = 0$, $\Gamma_H = \Gamma_0 = (1_\Gamma, \emptyset)$ consists of exactly one vertex 1_H. It follows that then $\forall i \in \{1, 2, 3, 4\}$, $C_i(\Gamma_H) = 0$, and so the lemma holds trivially.

Inductive step: We will assume that the lemma holds for the folding Γ_{n-1}, and show this implies the lemma is also true for $\Gamma_n = \Gamma_{n-1} \cup P_{n-1}$.

Put $u = s(P_{n-1})$ and $v = t(P_{n-1})$. If $u, v \notin V_{n-1}$, then by definition 2.4, u and v coincide, and (as a subgraph of Γ_n) P_{n-1} consists only of vertices of even degree. Thus, $d_3(\Gamma_n) = d_3(\Gamma_{n-1})$, and in particular, for $i = 1, \ldots, 4$, $C_i(\Gamma_n) = C_i(\Gamma_{n-1})$. By the inductive hypothesis, the lemma holds.

By definition 2.4 of directed trail decomposition, it only remains to consider the case when *both* $u, v \in V_{n-1}$. Note that all vertices in $V_n \backslash V_{n-1}$ must be of even degree (either 2 or 4) in Γ_n, and so do not factor into the 3-balancedness of Γ_n. For clarity, we shall denote a vertex $u \in V_{n-1}$ as u^+ whenever we are considering it as a vertex in V_n. For example, we denote the degree of a vertex u in Γ_{n-1} as $d(u)$, while denoting the degree of the same vertex in Γ_n as $d(u^+)$. Since Γ_{n-1} is a subgraph of Γ_n, it follows that $\{w \in V_{n-1} \mid d(w^+) > d(w)\} = \{u, v\}$. Let e_u [resp. e_v] be the first [resp. last] edge in P_{n-1}, and denote its label by l_u [resp. l_v], where $l_u, l_v \in \{a, b\}$. The various edges, vertices and labels are depicted below.



The case when $u = v$: If $u = v$, then it must be that $d(u) = 2$ and $d(u^+) = 4$. So, $d_3(\Gamma_n) = d_3(\Gamma_{n-1})$, and in particular, for $i = 1, \ldots, 4$, $C_i(\Gamma_n) = C_i(\Gamma_{n-1})$. By the inductive hypothesis, the lemma holds.

The case when $u \neq v$: Clearly $d(v^+) = d(v) + 1$ and $d(u^+) = d(u) + 1$. It follows that $d(u), d(v) \in \{2, 3\}$, and $d(u^+), d(v^+) \in \{3, 4\}$. We proceed now by considering each of the possible cases.

When $d(u) = 2$: Since u has degree 2, and Γ_{n-1} is the union of directed trails, u must have one incoming and one outgoing edge in Γ_{n-1}. But since $l_u \in \{a, b\}$, it must be that (in Γ_n) u^+ has 2 outgoing edges and 1 incoming edge, i.e. u^+ is either of type C_1 or C_3. Thus, the addition of the trail P_{n-1} caused the degree 2 vertex $u \in V_{n-1}$ to be transformed into a degree 3 vertex $u^+ \in V_n$ of type C_1 or type C_3. In summary, the transition $u \rightsquigarrow u^+$ causes the quantity $C_1 + C_3$ to increase by 1.

When $d(u) = 3$: Since vertex u^+ has degree 4 in Γ_n, and $l_u \in \{a, b\}$, the vertex u^+ has one more outgoing edge than u. It follows that in Γ_{n-1}, the vertex u had two incoming edges and one outgoing edge, i.e. u was either of type C_2 or C_4. Thus, the addition of trail P_{n-1} caused the degree 3 vertex $u \in V_{n-1}$ whose type was either C_2 or C_4 to be transformed into a degree 4 vertex $u^+ \in V_n$. In summary, the transition $u \rightsquigarrow u^+$ causes the quantity $C_2 + C_4$ to decrease by 1.

When $d(v) = 2$: Since v has degree 2, and Γ_{n-1} is the union of directed trails, v must have one incoming and one outgoing edge in Γ_{n-1}. But since $l_v \in \{a, b\}$, it must be that (in Γ_n) v^+ has 2 incoming edges and 1 outgoing edge, i.e. v^+ is either of type C_2 or C_4. Thus, the addition of the trail P_{n-1} caused the degree 2 vertex $v \in V_{n-1}$ to be transformed into a degree 3 vertex $v^+ \in V_n$ of type C_2 or type C_4. In summary, the transition $v \rightsquigarrow v^+$ causes the quantity $C_2 + C_4$ to increase by 1.

When $d(v) = 3$: Since vertex v^+ has degree 4 in Γ_n, and $l_v \in \{a, b\}$, the vertex v^+ has one more incoming edge than v. It follows that in Γ_{n-1}, the vertex v had two outgoing edges and one incoming edge, i.e. v was either of type C_1 or type C_3. Thus, the addition of trail P_{n-1} caused the degree 3 vertex $v \in V_{n-1}$ whose type was either C_1 or C_3 to be transformed into a degree 4 vertex $v^+ \in V_n$. In summary, the transition $v \rightsquigarrow v^+$ causes the quantity $C_1 + C_3$ to decrease by 1.

The conclusions of this analysis are summarized in the table below.

	$d(v) = 2$	$d(v) = 3$
$d(u) = 2$	$v \rightsquigarrow v^+$: $C_2 + C_4$ increases by 1 $u \rightsquigarrow u^+$: $C_1 + C_3$ increases by 1	$v \rightsquigarrow v^+$: $C_1 + C_3$ decreases by 1 $u \rightsquigarrow u^+$: $C_1 + C_3$ increases by 1
$d(u) = 3$	$v \rightsquigarrow v^+$: $C_2 + C_4$ increases by 1 $u \rightsquigarrow u^+$: $C_2 + C_4$ decreases by 1	$v \rightsquigarrow v^+$: $C_1 + C_3$ decreases by 1 $u \rightsquigarrow u^+$: $C_2 + C_4$ decreases by 1

The induction hypothesis $C_1(\Gamma_{n-1}) + C_3(\Gamma_{n-1}) = C_2(\Gamma_{n-1}) + C_4(\Gamma_{n-1})$ together with the table above shows that $C_1(\Gamma_n) + C_3(\Gamma_n) = C_2(\Gamma_n) + C_4(\Gamma_n)$. By induction on n then, the lemma holds.

□

REMARK 3.3. In [**10**], Walter Neumann showed that if $H, K \leqslant_{\text{f.g.}} F_2$ are a counterexample to the conjecture, then $\exists i \in \{1, 2, 3, 4\}$ s.t. $C_i(\Gamma_H) > \frac{1}{2} d_3(\Gamma_H)$ and $C_i(\Gamma_K) > \frac{1}{2} d_3(\Gamma_K)$. Clearly, if a group has a 3-balanced folding, then no more than half of its degree 3 vertices can be of the same type. Thus, it follows from W. Neumann's result that if H has a 3-balanced folding, then there is no $K \leqslant_{\text{f.g.}} F_2$ for which the pair (H, K) are a counterexample to the conjecture.

The remark that follows will be used to argue that a subsequent theorem about subgroups of F_2 holds in general for the finitely generated subgroups of any free group F.

REMARK 3.4. Take ϕ_n to be the homomorphism of $F_n = F(\{x_1, \ldots, x_n\})$ into $F_2 = F(\{a, b\})$ defined by $\phi_n : x_i \mapsto a^i b a^i$, $(i = 1, \ldots, n)$. It is easy to verify that ϕ_n is an embedding which takes positive words in F_n to positive words in F_2. In particular, if $H \leqslant_{\text{f.g.}} F_n$, then rank $\phi(H) =$ rank H, and if H has a positive generating set, then $\phi(H)$ has a positive generating set.

The main theorem may now be proved:

THEOREM 3.5. *If H, K are two finitely generated subgroups of a free group F and at least one of the two subgroups is generated by a set of positive words, then the pair (H, K) satisfy the Hanna Neumann conjecture.*

PROOF. WLOG, let H have a positive generating set

$$B_H = \{h_1, \cdots, h_n\}$$

Suppose first that $F = F_2$. By lemma 2.9, the existence of a positive generating set implies that Γ_H is strongly connected. By lemma 2.7, a directed trail decomposition of Γ_H exists. By lemma 3.2, Γ_H is 3-balanced. Finally, by remark 3.3, H cannot be part of any counterexample to the conjecture. This proves the case when $F = F_2$.

Now suppose $F \neq F_2$. Since H, K are finitely generated, WLOG, we can assume that $F = F_n$ for some finite n. If H, K were a counterexample to the conjecture, then by remark 3.4, $\phi_n(H), \phi_n(K) \leqslant_{\text{f.g.}} F_2$, $\phi_n(H)$ is positively generated, such that rank $H =$ rank $\phi_n(H)$, rank $K =$ rank $\phi_n(K)$, and

$$\text{rank } H \cap K = \text{rank } \phi_n(H \cap K) = \text{rank } (\phi_n(H) \cap \phi_n(K))$$

Thus $\phi_n(H), \phi_n(K)$ are a pair of positively generated subgroups of F_2 which are a counterexample to the conjecture. This contradicts our proof for the case when $F = F_2$. This proves the case when $F \neq F_2$. □

3.1. Further analysis. The proof of theorem 3.5 hinges on two key ideas: (i) foldings which have directed trail decompositions are necessarily 3-balanced, and (ii) finitely generated subgroups of F_2 which have 3-balanced foldings cannot be part of any counterexample to the conjecture. Our success in this approach naturally leads us to inquire about necessary and sufficient conditions for a folding to possess a directed trail decomposition. In this section, we answer this question for finitely generated subgroups of F_2.

Definition 3.6. A strongly connected component $\widetilde{J}$ is referred to as a **source** [resp. **sink**] if $\widetilde{J}$ consists of a single vertex v, where v has degree 2 in Γ, and v is incident to exactly two outgoing [resp. incoming] edges. A group $H \leqslant_{\text{f.g.}} F(X)$

is called **source/sink-free** (with respect to basis X) if Γ_H contains neither source nor sink vertices.

The next lemma describes a local structural property of a folding which is equivalent to its having a directed trail decomposition.

Lemma 3.7. *Let $H \leqslant_{f.g.} F_2$ and $\Gamma = (V, E)$ be the folding of H. Then Γ has no sources and no sinks if and only if Γ has a directed trail decomposition.*

Proof. $\Rightarrow$ Decompose Γ into strongly connected components. Let $\widetilde{J}_1, \ldots, \widetilde{J}_m$ denote those strongly connected components whose size (number of vertices) is > 1. Take

$$G_0 = \bigcup_{i=1,\ldots,m} \widetilde{J}_i$$

Since each $\widetilde{J}_i$ is strongly connected, by lemma 2.7, each $\widetilde{J}_i$ has a directed trail decomposition Q_i. Since the $\widetilde{J}_i$ are pairwise disjoint, it follows that G_0 has a directed trail decomposition P_0—simply take P_0 to be $Q_1, \ldots, Q_m$. We extend P_0 to a directed trail decomposition of Γ in stages. At each successive stage i (starting at $i = 0$):

(1) If $E[\Gamma] \backslash E[G_i] = \emptyset$, halt. Otherwise, select an edge $(u, v) \in E[\Gamma] \backslash E[G_i]$. Starting at vertex u we walk backwards along (arbitrarily chosen) incoming edges, until reach a vertex $u' \in V[G_i]$. Likewise, starting at vertex v we walk forwards along (arbitrarily chosen) outgoing edges until we reach a vertex $v' \in V[G_i]$. We cannot get stuck in either of these steps, because Γ contains neither sources nor sinks; we cannot get trapped in a loop before we find a vertex in $V[G_i]$ because then we have discovered a strongly connected component that must have been omitted from the set $\widetilde{J}_1, \ldots \widetilde{J}_m$, a contradiction.

Define P_{i+1} to be the directed trail from $u' \rightsquigarrow u \to v \rightsquigarrow v'$ described above. Notice that P_{i+1} attaches G_i at precisely its endpoints u', v'. We append P_{i+1} to the directed trail decomposition at stage i, obtaining a trail decomposition of $G_{i+1} = G_i \cup P_{i+1}$. Increment i, then repeat step 1.

At the end of this procedure, we have constructed a directed trail decomposition of Γ, as claimed.

$\Leftarrow$ Suppose that $\Gamma = (V, E)$ has a directed trail decomposition $P_0, \ldots, P_k$ and (towards contradiction) also has a source vertex $v_0 \in V$. Since $\{P_i \mid i = 0, \ldots, k\}$ covers $E[\Gamma]$, let $i_0 \in \{0, 1, \ldots, k\}$ be the least integer for which $v_0 \in V[P_{i_0}]$.

If $v_0 \neq s(P_{i_0}), t(P_{i_0})$ then v_0 is an intermediate vertex of P_{i_0} which has two outgoing edges, contradicting the fact that P_{i_0} is a directed trail. If $v_0 \in \{s(P_{i_0}), t(P_{i_0})\}$ then minimality of i_0 implies that

$$V[P_{i_0}] \cap \bigcup_{j=0}^{i_0-1} V[P_j] = \emptyset$$

so by definition 2.4 of directed trail decomposition, the endpoints of P_{i_0} must coincide. It follows that the final edge in P_{i_0} is not oriented towards $t(P_{i_0})$, contradicting the fact that P_{i_0} is a directed trail. Thus, no such source vertex v_0 exists.

A completely analogous argument shows if Γ has a sink vertex, then Γ cannot have a directed trail decomposition. $\square$

Lemma 3.7 then has the following consequence for the conjecture:

THEOREM 3.8. *If H, K are two finitely generated subgroups of the free group F_2 and at least one of the two subgroups is source/sink-free, then the pair (H, K) satisfy the Hanna Neumann conjecture.*

PROOF. If Γ_H has neither source nor sink vertices, then by lemma 3.7, Γ_H has a directed trail decomposition. So, lemma 3.2 applies and hence Γ_H must be 3-balanced. Then, by remark 3.3, H cannot be part of any counterexample to the conjecture. $\square$

3.2. Remarks on invariants. The properties of being source/sink-free and positively generated depend on the choice of basis for the ambient free group F. This leads us to define the following natural basis-invariant versions of these properties:

DEFINITION 3.9. A subgroup $H \leqslant_{\text{f.g.}} F_2 = F(\{a, b\})$ is termed **potentially source/sink-free** if $\exists \phi \in Aut(F_2)$, such that $\Gamma_{\phi(H)}$ is source/sink-free (with respect to basis $\{a, b\}$).

DEFINITION 3.10. A subgroup $H \leqslant_{\text{f.g.}} F(X)$ is called **potentially positive** if for some $\phi \in Aut(F)$, $\phi(H)$ is positively generated (with respect to basis X).

Theorems 3.5 and 3.8 then have the following corollaries:

COROLLARY 3.11. If H, K are two finitely generated subgroups of a free group F and at least one of the two subgroups is potentially positive, then the pair (H, K) satisfy the Hanna Neumann conjecture.

COROLLARY 3.12. If H, K are two finitely generated subgroups of the free group F_2 and at least one of the two subgroups is potentially source/sink-free, then the pair (H, K) satisfy the Hanna Neumann conjecture.

4. Algorithms and computational complexity

In this section, we will show that it is decidable whether a finitely generated subgroup of a free group is positively generated. Specifically, we show

THEOREM 4.1. *Let $X = \{x_1, \ldots, x_m\}$ and $F_m = F(X)$. There exists an algorithm which for any finite set of freely reduced words $S = \{h_1, \ldots, h_n\} \subset F_m$ determines whether $H = \langle S \rangle$ is positively generated, and if so outputs a positive basis for H. This algorithm operates in $O(m^2\ell^3 \log \ell)$ time, where $\ell = \sum_{i=1}^{n} |h_i|$. If additionally $h_1, h_2, \ldots, h_n$ are known to form a Nielsen reduced set, then the algorithm may be modified to terminate in $O(\ell^2)$ time.*

We shall prove theorem 4.1 by describing an algorithm which operates in three phases and achieves the stated claims. In the first phase, S is transformed into a folding Γ. In the second phase, the algorithm determines if H is positively generated by testing whether Γ is strongly connected. If so, then in the third phase, a positive basis for H is computed using a directed trail decomposition of Γ. Throughout the computation, Γ is stored as a transition table T mapping $V \times X^{\pm 1} \to 2^V$. For each $u \in V$ and $c \in X$, the set $T(u, c)$ is stored as a balanced binary tree (e.g. as a splay tree [**14**]) so as to permit insertion and deletion of elements in $O(\log |T(u, c)|)$ time.

4.1. Phase I: Building the folding for H.

In the first phase the algorithm constructs the folding Γ_H by applying procedure 1 listed below, which in turn makes use of the sub-procedure `FOLD`.

Procedure 1 BUILD-FOLDING$(h_1, h_2, \ldots, h_n)$

1: $V \leftarrow \{1\}$
2: $E \leftarrow \emptyset$
3: $\Gamma \leftarrow (V, E)$
4: **for all** h in $\{h_1, h_2, \ldots, h_n\}$ **do**
5: Add a directed loop to Γ, starting and ending at 1, labelled by h_i
6: **end for**
7: `FOLD`(1)
8: `Output` Γ

LEMMA 4.2. *Procedure* `BUILD-FOLDING` *runs in $O(m^2\ell^3 \log \ell)$ time.*

PROOF. The time required for lines 1-6 is $O(\ell)$. It remains to consider line 7. Note that lines 3-14 of procedure `FOLD` collapse the vertices of $T(u, c)$ to a single vertex v; clearly no more than $|V|$ executions of lines 3-14 may occur. During a single execution of lines 3-14, nested loop variables w, d, and q take at most $|V|$, m, and $|V|$ values, respectively. Since each element in $Im(T)$ is a balanced binary tree of size at most $|V|$, the insertion and deletion operations that need to be performed in lines 6 and 8 require at most $O(\log |V|)$ factor overhead. It follows that a single execution of lines 3-14 takes $O(m|V|^2 \log |V|)$ time. Taking into account the outermost loop (lines 1,16) we see the execution of line 7 in procedure `BUILD-FOLDING` takes at most $O(m^2|V|^3 \log |V|) = O(m^2\ell^3 \log \ell)$ time. This gives us an upper bound on the time complexity of `BUILD-FOLDING`. $\square$

Procedure 2 FOLD(u)

for all c in $X^{\pm 1}$ **do**

if $|T(u,c)| > 1$ **then**

$v \leftarrow$ any vertex in $T(u,c)$

for all w in $T(u,c) \backslash \{v\}$ **do**

for all d in $X^{\pm 1}$ **do**

$T(v,d) \leftarrow T(v,d) \cup T(w,d)$

for all q in $T(w,d)$ **do**

$T(q,d^{-1}) \leftarrow T(q,d^{-1}) \cup \{v\} \backslash \{w\}$

end for

$T(w,d) \leftarrow \emptyset$

end for

Delete w from V

end for

$T(u,c) \leftarrow \{v\}$

`FOLD`(v)

end if

end for

If the generating set $S = \{h_1, h_2, \ldots, h_n\}$ is *known* to be Nielsen reduced (see [**8**, pages 6-9]), then the next lemma shows that the folding can be built significantly faster using a different procedure, `BUILD-NIELSEN-FOLDING`, listed below.

Procedure 3 BUILD-NIELSEN-FOLDING$(h_1, h_2, \ldots, h_n)$

$V \leftarrow \{1\}$

$E \leftarrow \emptyset$

for all h_i in $\{h_1, h_2, \ldots, h_n\}$ **do**

Read h_i in Γ starting at vertex 1, and thereby obtain a decomposition $h_i = p_i \circ q_i$, where p_i is the maximal length prefix of h_i which can be read in Γ starting at vertex 1. Let u_i be the vertex reached after reading p_i

Read q_i^{-1} in Γ starting at vertex 1, and so obtain a decomposition $q_i^{-1} = r_i \circ s_i$, where r_i is the maximal length prefix of q_i^{-1} which can be read in Γ starting at vertex 1. Let v_i be the vertex reached after reading r_i.

if $u_i \neq v_i$ **then**

Add a chain (labelled by s_i^{-1}) connecting u_i to v_i. Augment V, E, and T appropriately.

else

Decompose $s_i = t_i \circ c_i \circ t_i^{-1}$ where c_i is cyclically reduced. Add a spur labelled t_i starting at u_i. At the end of this spur, attach a loop labelled c_i.

end if

end for

`Output` Γ

LEMMA 4.3. *Procedure* `BUILD-NIELSEN-FOLDING` *runs in* $O(\ell)$ *time.*

PROOF. Since S is a Nielsen reduced set, at each iteration of lines 4-5, h_i is Nielsen reduced with respect to $\{h_1, \ldots, h_{i-1}\}$. It follows that s_i is always a non-trivial word. If u_i and v_i are distinct, then we add a chain connecting u_i to v_i and label this by s_i^{-1} (line 7). If u_i and v_i coincide, then s_i may fold into itself. This cancellation can be predicted, however, by decomposing s_i as a product $t_i \circ c_i \circ t_i^{-1}$ where c_i is cyclically reduced (line 9). The time required to perform lines 4-10 of procedure `BUILD-NIELSEN-FOLDING` is $O(|h_i|)$, so the lemma follows. □

4.2. Phase II: Determining if H is positively generated. Given the folding Γ constructed in phase I, we may apply Tarjan's algorithm [**18**] to determine whether Γ is strongly connected. Tarjan's algorithm is based on a modified depth-first search (see [**3**]) and operates in $O(|V| + |E|)$ time. By lemma 2.10, this is equivalent to determining whether H is a positively generated subgroup of F_m. Since $|V| + |E|$ is bounded above by 2ℓ, it follows that

LEMMA 4.4. *There is an effective procedure which given the folding of a subgroup* $H = \langle\{h_1, \ldots, h_n\}\rangle \leqslant F_m$, *decides whether* H *is positively generated in* $O(\ell)$ *time, where* $\ell = \sum_{i=1}^{n} |h_i|$.

4.3. Phase III: Computing a positive basis for H. If in phase II we determine that Γ is a strongly connected directed graph, we proceed to compute a positive basis for H. The proof of lemma 2.7 describes one procedure which constructs a strong directed trail decomposition for a strongly connected graph. A more efficient approach would be to use a modification of the standard algorithm for computing the "open ear decomposition" of a 2-vertex connected *undirected* graph (see for example [**12**, pages 276-286]). These algorithms run in $O(|V| + |E|)$ time, and can be modified to compute directed trail decompositions of strongly connected directed graphs.

Next, we follow the procedure outlined in the proof of lemma 2.10, which describes how to compute a positive basis for a subgroup of a free group, given a strong directed trail decomposition $P_0, \ldots, P_n$ for its folding: We consider the graphs $\Gamma_0 = (1_\Gamma, \emptyset)$ and $\Gamma_i = \bigcup_{j=0}^{i-1} P_j$ (for $i = 1, \ldots n+1$). Since Γ_i is strongly connected, we can use 2 applications of breadth-first search to compute a directed trail s_i in Γ_i, from 1_H to $s(P_i)$, and a directed trail t_i from $t(P_i)$ to 1_H. If we denote the words read along P_i, s_i, and t_i as $w_{P_i}, w_{s_i}, w_{t_i}$ respectively, then the algorithm outputs $h_i = w_{s_i} \circ w_{P_i} \circ w_{t_i}$ for $i = 0, \ldots, n$. The time required is dominated by the $2(n+1)$ executions of breadth-first search used to compute the s_i and t_i. Since each breadth-first search takes $O(|V| + |E|) = O(\ell)$ time, the total time required is $O(n\ell)$ which is $O(\ell^2)$. We have shown

LEMMA 4.5. *There is an effective procedure which given the strongly connected folding of a positively generated subgroup* $H = \langle\{h_1, \ldots, h_n\}\rangle \leqslant F_m$, *outputs a positive basis for* H *in* $O(\ell^2)$ *time, where* $\ell = \sum_{i=1}^{n} |h_i|$.

Theorem 4.1 now follows.

PROOF. (Theorem 4.1) The algorithm operates in the three phases described above. In the case where S is known to be a Nielsen reduced set, lemmas 4.3 and 4.4 show that phases I and II take $O(\ell)$ time, while phase III dominates, taking

$O(\ell^2)$ time by lemma 4.5. If S is not known to be Nielsen reduced then phase I dominates, requiring $O(m^2\ell^3 \log \ell)$ time by lemma 4.2, while phases II and III take $O(\ell)$ and $O(\ell^2)$ time respectively. $\square$

5. Open questions

We have seen that there are are efficient algorithms to determine whether a subgroup $H \leqslant_{\text{f.g.}} F_m$ is positively generated. We ask whether there is an algorithm to decide whether a given subgroup is *potentially positive* (see definition 3.10) ? In remark 2.13, we noted that the class of positively generated subgroups of a free group is not closed under intersections. We ask whether the class of *potentially-positive* subgroups is closed under intersections?

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