Computation with Polytopal Uncertainty in d-Dimensional Space

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1 Introduction

A measurement is a quantitative description of the transformation required to carry one state to another state. For example, measuring the position of an object involves determining what translation and rotations that will move the axes at the origin to the object's present configuration. In an ideal world, making a measurement between two states u and v yields precisely one transformation. In such a setting, information can be easily consolidated: a measurement from u to v can be composed with a measurement from v to w—all that is needed is vector algebra. In the real world, however, errors and limitations in the sensitivities of the measurement apparatus make it impossible to precisely determine the transformation that will carry u to v. At best, we can determine a small set of candidates transformations, and assert that one of these carries u to v. We accept that the size of this set of candidates will never be 1, or equivalently the "uncertainty" in our measurement will never be zero. In this situation, naive vector algebra does not suffice to allow us to compose measurements. The process by which we consolidate information from several measurements poses a difficult computational problem. We will describe and address several aspects of the problem in this paper.

The principal applications of these results lie in automated consolidation of information derived from sensor measurements in robot planning [5] and in automated registration in augmented reality systems [4]. In recent years, many approaches to this research area have shied away from analytic solutions, and favored the use of Bayesian networks [1, 3] and other statistical schemes [6, 2] to consolidate measurements and minimize uncertainty in robot planning. One notable exception is the work of Rajan and Taylor which provided closed form solutions [7] in low-dimensional settings.

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2 Polytopal Error Model

The *states* we consider are those of rigid objects in d-dimensional space, represented by their position and orientation. We are considering state space so d may be quite large (and in particular, bigger than 3). For example, we might take $d = 5$ to consist of the three spatial coordinates, the temperature, and the luminosity of an object. The abstract notion of the object's orientation represents a privileged direction, for instance, the gradient vector along which the object is most likely to move in its state space.

Within such a model, position is specified as a d-dimensional vector p quantifying the object's offset from the origin, and orientation is represented by a orthogonal determinant $+1$ matrix R. Since $SO(d)$ is a real compact Lie group of dimension $k(d) \stackrel{def}{=} \frac{d(d-1)}{2}$, the state of a rigid d-dimensional object can be encoded as a vector u in $X_d \stackrel{def}{=} \mathbb{R}^{d+k(d)}$.

Next, we specify the types of *transformations* we shall consider. A transformation on X_d is an ordered pair $T = (p, R)$ where p is a translation and R is a rotation (in $\mathbb{R}^d$). Note that since p can be specified using a d-dimensional vector, and R is an orthogonal matrix from $SO(d)$, it follows that a transformation (p, R) can also be represented as a vector in X_d.

Given states u, v in X_d, a transformation (p^*_{uv}, R^*_{uv}) is said to *carry* u *to* v if

$$p^*_{uv} + R^*_{uv} u \quad = \quad v.$$

Because our measurement apparatus is error-prone and has a limited sensitivity, it is impossible to determine an exact transformation (p^*_{uv}, R^*_{uv}) which will carry u to v. We represent a *measurement from* u *to* v as an ordered pair

$$((p^0_{uv}, R^0_{uv}), E_{uv})$$

where (p^0_{uv}, R^0_{uv}) is a nominal transformation and $E_{uv} \subset X_d$ is a set of perturbative transformations. To make the measurement is to assert that *there exists some* perturbative transformation in E_{uv}, which when composed with the nominal transformation (p^0_{uv}, R^0_{uv}), carries u to v.

In this paper, we consider a model of uncertainty where the set of perturbative transformations E_{uv} is a polytopal subset of X_d, specified by a set of linear inequalities. Thus we follow the uncertainty model postulated by Rajan and Taylor in [7] in favor of other candidates, e.g. ellipsoidal models, probabilistic models [3], etc.

Given a vector $\sigma = (\epsilon_0, \cdots, \epsilon_{d-1}, \alpha_0, \ldots, \alpha_{k(d)-1})^T$ in E_{uv}, we interpret σ as encoding the perturbative transformation $(p_\sigma, I + R_\sigma)$, where

$$p_\sigma \stackrel{def}{=} \begin{pmatrix} \epsilon_0 \\ \vdots \\ \epsilon_{d-1} \end{pmatrix}$$

and

$$R_\sigma \stackrel{def}{=} \begin{pmatrix} 0 & -\alpha_0 & -\alpha_1 & \cdots & & \\ \alpha_0 & 0 & -\alpha_2 & \cdots & & \\ \alpha_1 & \alpha_2 & 0 & & & \vdots \\ \vdots & \vdots & \vdots & 0 & & -\alpha_{k(n)-2} \\ & & & & 0 & -\alpha_{k(n)-1} \\ & & \cdots & \alpha_{k(n)-2} & \alpha_{k(n)-1} & 0 \end{pmatrix}.$$

Formally then, $((p^0_{uv}, R^0_{uv}), E_{uv})$ is called *a measurement from u to v* if

$$\begin{aligned} v &= p^0_{uv} + R^0_{uv}(p_{\sigma_{uv}} + (I + R_{\sigma_{uv}})u) \\ &= p^0_{uv} + R^0_{uv} p_{\sigma_{uv}} + R^0_{uv} u + R^0_{uv} R_{\sigma_{uv}} u \end{aligned} \quad (1)$$

for some $\sigma_{uv} \in E_{uv}$.

3 Results

Consider the composition of two measurements in the afforementioned model, the first carrying u to v, and the second carrying v to w. Although each individual measurement has an associated polytopal set of perturbative transformations, the composition of two measurements exhibits higher order (quadratic) surfaces. In general, long sequences of compositions lead to even higher order objects which are difficult to manipulate analytically. It is therefore crucial to devise a good polytopal approximation of the set of perturbative transformations of a composite measurement. A similar problem also arises when we consider the inverse of a measurement, as this is also, generally not be a (linear) polytope. *The principal contribution of this paper is the derivation of polytopal approximations for the composition and inversion of measurements under the polytopal bounded model of error.* Rajan and Taylor [7] provided analogous solutions for the restricted case when $d = 3$. In this paper, more general techniques are developed to extend their results to arbitrary dimensions. In providing such approximations, we close the class of all d-dimensional measurements under the operation of composition and inversion.

3.1 Composition of Measurements

Let $((p^0_{uv}, R^0_{uv}), E_{uv})$ be a measurement from u to v, and $((p^0_{vw}, R^0_{vw}), E_{vw})$ be a measurement from v to w. Then the formal definition of measurement implies that there exist $\sigma_{uv} \in E_{uv}$ and $\sigma_{vw} \in E_{vw}$ such that

$$\begin{aligned} w &= p^0_{vw} + R^0_{vw}(p_{\sigma_{vw}} + (I + R_{\sigma_{vw}})(p^0_{uv} + R^0_{uv}p_{\sigma_{uv}} + R^0_{uv}u + R^0_{uv}R_{\sigma_{uv}}u)) \\ &= p^0_{vw} + R^0_{vw}p_{\sigma_{vw}} + \\ &\quad R^0_{vw}p^0_{uv} + R^0_{vw}R^0_{uv}p_{\sigma_{uv}} + R^0_{vw}R^0_{uv}u + R^0_{vw}R^0_{uv}R_{\sigma_{uv}}u + \\ &\quad R^0_{vw}R_{\sigma_{vw}}p^0_{uv} + R^0_{vw}R_{\sigma_{vw}}R^0_{uv}u + \\ &\quad R^0_{vw}R_{\sigma_{vw}}R^0_{uv}p_{\sigma_{uv}} + R^0_{vw}R_{\sigma_{vw}}R^0_{uv}R_{\sigma_{uv}}u. \end{aligned}$$

The last two terms are second-order, and may be considered infinitesimal if the perturbative transformations are small relative to the nominal transformation. Using this fact and rearranging terms, we see that

$$\begin{aligned} w &\approx p^0_{vw} + R^0_{vw}p^0_{uv} + \\ &\quad R^0_{vw}R^0_{uv}p_{\sigma_{uv}} + R^0_{vw}p_{\sigma_{vw}} + R^0_{vw}R_{\sigma_{vw}}p^0_{uv} + \\ &\quad R^0_{vw}R^0_{uv}u + \\ &\quad R^0_{vw}R^0_{uv}R_{\sigma_{uv}}u + R^0_{vw}R_{\sigma_{vw}}R^0_{uv}u. \end{aligned}$$

Now suppose $((p^0_{uw}, R^0_{uw}), E_{uw})$ is a measurement from u to w. Then

$$w = p^0_{uw} + R^0_{uw}p_{\sigma_{uw}} + R^0_{uw}u + R^0_{uw}R_{\sigma_{uw}}u.$$

But if $((p^0_{uw}, R^0_{uw}), E_{uw})$ approximates the composition of our two measurements $((p^0_{uv}, R^0_{uv}), E_{uv})$ and $((p^0_{vw}, R^0_{vw}), E_{vw})$, then it must be that

$$\begin{aligned} p^0_{uw} &= p^0_{vw} + R^0_{vw}p^0_{uv} && (2) \\ R^0_{uw}p_{\sigma_{uw}} &= R^0_{vw}R^0_{uv}p_{\sigma_{uv}} + R^0_{vw}p_{\sigma_{vw}} + R^0_{vw}R_{\sigma_{vw}}p^0_{uv} && (3) \\ R^0_{uw}u &= R^0_{vw}R^0_{uv}u && (4) \\ R^0_{uw}R_{\sigma_{uw}}u &= R^0_{vw}R^0_{uv}R_{\sigma_{uv}}u + R^0_{vw}R_{\sigma_{vw}}R^0_{uv}u && (5) \end{aligned}$$

Assuming $u \neq 0$, equation (4) above implies

$$R^0_{uw} = R^0_{vw}R^0_{uv} \tag{6}$$

and so (3) and (5) reduce (respectively) to

$$\begin{aligned} p_{\sigma_{uw}} &= p_{\sigma_{uv}} + (R^0_{uv})^{-1}p_{\sigma_{vw}} + (R^0_{uv})^{-1}R_{\sigma_{vw}}p^0_{uv}. && (7) \\ R_{\sigma_{uw}} &= R_{\sigma_{uv}} + (R^0_{uv})^{-1}R_{\sigma_{vw}}R^0_{uv} && (8) \end{aligned}$$

Expressions (2) and (6) indicate how to construct the nominal transformation of the composite measurement from the nominal transformations of the constituent measurements. Expressions (7) and (8) indicate how to reconstruct σ_{uw} as a linear function of σ_{uv} and σ_{vw}, showing that we have successfully approximated the set of perturbative transformations in the composite measurement by a (linear) polytope.

3.2 Inversion of Measurements

Starting with equation (1) and rearranging, we get

$$(I + R_{\sigma_{uv}})^{-1}(R^0_{uv})^{-1}\left[v - p^0_{uv} - R^0_{uv}p_{\sigma_{uv}}\right] \quad = \quad u. \tag{9}$$

Since $(I + R_{\sigma_{uv}})$ is orthogonal, $(I + R_{\sigma_{uv}})^{-1} = I - R_{\sigma_{uv}}$. It follows that

$$\begin{aligned} u &= (I - R_{\sigma_{uv}})(R^0_{uv})^{-1}\left[v - p^0_{uv} - R^0_{uv}p_{\sigma_{uv}}\right] \\ &= -(R^0_{uv})^{-1}p^0_{uv} \\ &\quad -(R^0_{uv})^{-1}R^0_{uv}p_{\sigma_{uv}} + R_{\sigma_{uv}}(R^0_{uv})^{-1}p^0_{uv} \\ &\quad +(R^0_{uv})^{-1}v \\ &\quad -R_{\sigma_{uv}}(R^0_{uv})^{-1}v \\ &\quad +R_{\sigma_{uv}}(R^0_{uv})^{-1}R^0_{uv}p_{\sigma_{uv}}. \end{aligned}$$

The last term is second-order and may be considered infinitesimal if the perturbative transformations are small relative to the nominal transformation. Now suppose $((p^0_{vu}, R^0_{vu}), E_{vu})$ in a measurement from v to u. Then

$$u \quad = \quad p^0_{vu} + R^0_{vu}p_{\sigma_{vu}} + R^0_{vu}v + R^0_{vu}R_{\sigma_{vu}}v.$$

But if $((p^0_{vu}, R^0_{vu}), E_{vu})$ approximates the inverse of $((p^0_{uv}, R^0_{uv}), E_{uv})$, then it must be that

$$p^0_{vu} \quad = \quad -(R^0_{uv})^{-1}p^0_{uv} \tag{10}$$
$$R^0_{vu}p_{\sigma_{vu}} \quad = \quad -(R^0_{uv})^{-1}R^0_{uv}p_{\sigma_{uv}} + R_{\sigma_{uv}}(R^0_{uv})^{-1}p^0_{uv} \tag{11}$$
$$R^0_{vu}v \quad = \quad (R^0_{uv})^{-1}v \tag{12}$$
$$R^0_{vu}R_{\sigma_{vu}}v \quad = \quad -R_{\sigma_{uv}}(R^0_{uv})^{-1}v. \tag{13}$$

Assuming $v \neq 0$, equation (12) above implies

$$R^0_{vu} \quad = \quad (R^0_{uv})^{-1} \tag{14}$$

and so (11) and (13) reduce to

$$p_{\sigma_{vu}} \quad = \quad -R^0_{uv}(R^0_{uv})^{-1}R^0_{uv}p_{\sigma_{uv}} + R^0_{uv}R_{\sigma_{uv}}(R^0_{uv})^{-1}p^0_{uv} \tag{15}$$
$$R_{\sigma_{vu}} \quad = \quad -R^0_{uv}R_{\sigma_{uv}}(R^0_{uv})^{-1}. \tag{16}$$

Expressions (10) and (14) indicate how to construct the nominal transformation of the inverse measurement from the nominal transformations of the original measurement. Expressions (15) and (16) indicate how to reconstruct σ_{vu} as a linear function of σ_{uv}, showing that we have successfully approximated the set of perturbative transformations in the inverse measurement by a (linear) polytope.

4 Conclusion and Future Work

In this paper, we derived linear approximations for composition and inversion of measurements in spaces of arbitrary dimension. By doing so, we close the class of d-dimensional measurements in the polytopal error model under the operations of composition and inversion. The derivation of these closed form expressions opens the door to many applications in automated consolidation of information derived from sensor measurements in robot planning as well as in automated registration in augmented reality systems. In future, we intend to investigate applications of these results in the design of proactive measurement and sensing strategies which seek to minimize uncertainty in robot motion planning.

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