Graphic Arithmetic II:
$+$ Irreducibility, Canonical Decompositions and Cancellation Laws

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Abstract

In this paper, we continue with our investigation of natural arithmetic properties on the set of all flow graphs, that is, the set of all finite directed connected multigraphs having a pair of distinguished vertices(see [1]). We introduce a graph operation called *-Deletion and study some of its properties. This leads to a notion of splitting vertex and w-Splitting of a flow graph. We study some properties of splitting vertices in a flow graph and show that a flow graph is $+$ reducibile if and only if it has a splitting vertex. We define a concept of splitting vertex ranking and use it to develop a theory of canonical + decompositions.

1 Introduction

In [1], we generalized classical arithmetic defined over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, to the set F consisting of all *flow graphs*: finite directed connected multigraphs [1] in which a pair of distinguished vertices is designated as the *source* and *target* vertex. The proposed model exhibits the property that the natural numbers appear as a submodel, with the directed path of length n playing the role of the standard integer n. We discussed basic features including associativity, distributivity, and various identities relating the order relation to addition and multiplication.

In this section, we review some definitions and results from [1].

Definition 1.1 (Flow graph)**.** *We define a* flow graph A *to be a triple* (G_A, s_A, t_A)*, where* G_A *is a finite*[2] *directed connected multigraph and* $s_A, t_A \in V[G_A]$ *are called the* source *and the*

[1] By multigraph we mean graphs in which parallel and loop edges are permitted.
[2] In this paper, we focus on finite flow graphs, although many of our results continue to hold in formulation which considers infinite flow graphs as well.

target *vertex of* A*, respectively. The set of all flow graphs is denoted* F*. The unique flow graph for which* $|V[G_A]| = 1$ *and* $|E[G_A]| = 0$ *is called* the trivial flow graph*; all other flow graphs are considered* non-trivial*. Given two flow graphs* $A = (G_A, s_A, t_A)$ *and* $B = (G_B, s_B, t_B)$*, a map* $\phi : A \to B$ *is said to* copy A *into* B *if (as a graph embedding)* ϕ *maps* G_A *injectively into* G_B *and additionally satisfies* $\phi(s_A) = s_B$*,* $\phi(t_A) = t_B$*. Flow graphs* A *and* B *are considered isomorphic if there is an injective morphism* $\phi : A \to B$ *for which* $Im(\phi) = B$*.*

In considering multigraphs with loops, we say that two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are *parallel* if $\{u_1, v_1\} = \{u_2, v_2\}$ are equal *as sets*. An edge $e = (u, v)$ is called a *loop edge* if u and v coincide.

Definition 1.2 (Trivial flow graph)**.** *A* flow graph $A = (G_A, s_A, t_A)$ *is called* the trivial flow graph *if* $|V[G_A]| = 1$ *and* $|E[G_A]| = 0$*. All other flow graphs are considered* non-trivial.

Definition 1.3. *Given any flow graph* A*, let* A' *be the flow graph obtained by swapping the source and the target of* A*.*

Definition 1.4 (Reflective flow graphs)**.** *A* flow graph $A = (G_A, s_A, t_A)$ *is called an* reflective *flow graph if* $A = A'$*. The set of all reflective flow graphs is denoted* $\mathcal{H}$*.*

Definition 1.5 (Infinitesimal flow graphs)**.** *A* flow graph $A = (G_A, s_A, t_A)$ *is called an* infinitesimal *flow graph if* $s_A = t_A$*. The set of all infinitesimal flow graphs is denoted* $\mathcal{I}$*. Note that an infinitesimal flow graph is necessarily reflective. The converse is false as the reflective example in Figure 1 shows.*

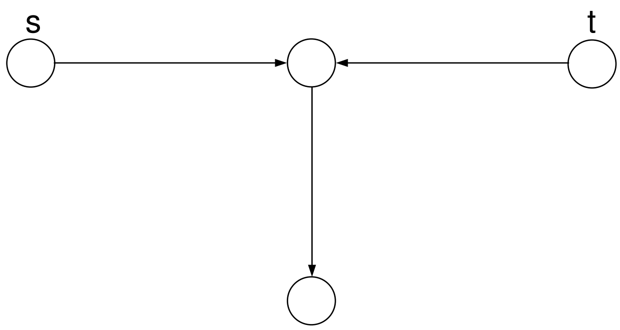

Figure 1: A non-infinitesimal flow graph in $\mathcal{H}$.

Definition 1.6. *Given any flow graph* A*, let* A^* *be the flow graph obtained by reversing all the arrows of* A*.*

Definition 1.7 (Reversible flow graphs)**.** *A* flow graph $A = (G_A, s_A, t_A)$ *is called a* reversible *flow graph if* $A = A^*$*. The set of all reversible flow graphs is denoted* $\mathcal{J}$*. Note that if for all vertices* u, v *in* V_A *we have* $\sharp(u, v) = \sharp(v, u)$*, then* A *is necessarily reversible. The converse is false as the reversible example in Figure 2 shows.*

Definition 1.8 (Self-conjugate flow graphs)**.** *A* flow graph $A = (G_A, s_A, t_A)$ *is called an* self-conjugate[3] *flow graph if* $A = A'^* = A^{*\prime}$*.*

[3]The motivation for the term *self-conjugate* will be clarified later, in item 4 of Section **??**.

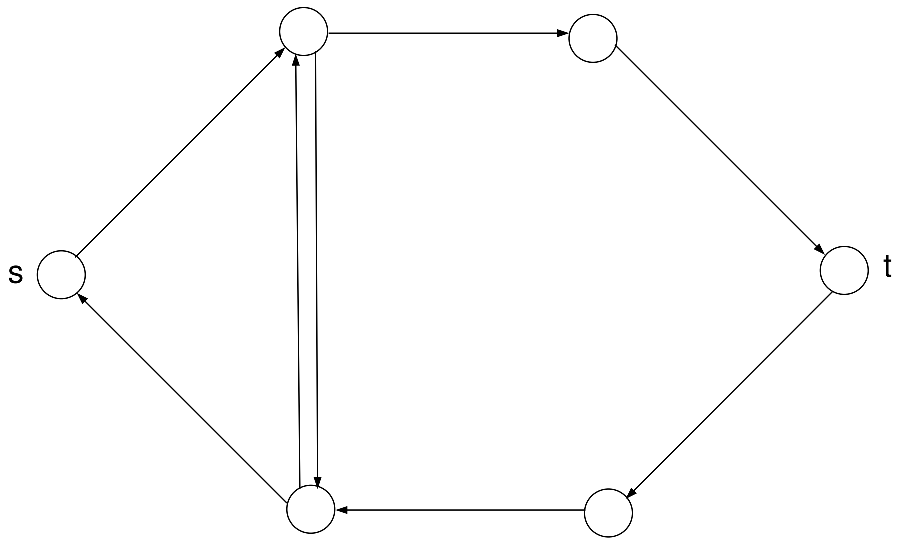

Figure 2: A flow graph in $\mathcal{J}$ having a non-symmetric adjacency matrix.

The set of all self-conjugate flow graphs is denoted $\mathcal{K}$. Note that if a flow graph is both reflective and reversible, it is necessarily self-conjugate. The converse is false as the self-conjugate example in Figure 3 shows.

Definition 1.9. *The* **rose** with n petals *is defined to be the infinitesimal flow graph R_n having one vertex and n loop edges. Roses R_1, R_2, R_3 are shown in the bottom left panel of Figure 4.*

Definition 1.10. *The* **star (antistar)** with n edges *is defined to be the infinitesimal flow graph S_n (S_n^*) having $n+1$ vertices $v_1, v_2, \ldots, v_n$ and $u = s = t$, with n edges from u to v_i (v_i to u) for each $i = 1, \ldots, n$. Stars S_1, S_2 and S_3 are shown in the bottom center panel of Figure 4, while anti-stars S_1^*, S_2^* and S_3^* are shown on the bottom right panel.*

Definition 1.11 (Graphical natural number)**.** *We represent the natural number n as a directed chain of length n, having $n+1$ vertices. More formally, let P_n be a directed chain of length n (having $n+1$ vertices) where each vertex has in-degree $\leqslant 1$ and out-degree $\leqslant 1$. Denote by s_n, the unique vertex in P_n having in-degree 0, and let t_n be the unique vertex in P_n having out-degree 0. The flow graph $F_n = (P_n, s_n, t_n)$ is referred* the graphic natural number n. *Define the map $i : \mathcal{N} \to \mathcal{F}$ as*

$$i : n \quad \mapsto \quad F_n.$$

We denote F_0 as 0 and F_1 as 1. Graphical natural numbers F_1, F_2 and F_3 are shown in the top left panel of Figure 4.

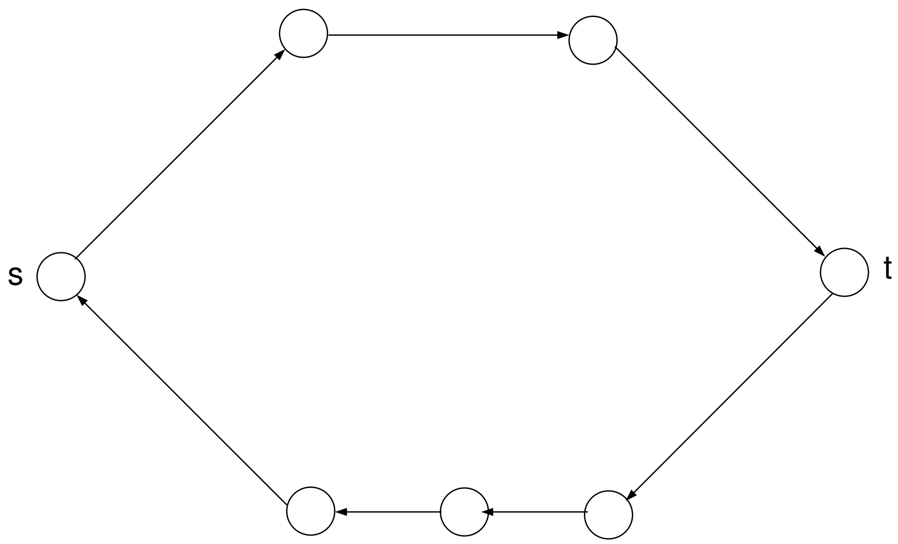

Figure 3: A flow graph in $\mathcal{K}$ that is neither reflective nor reversible.

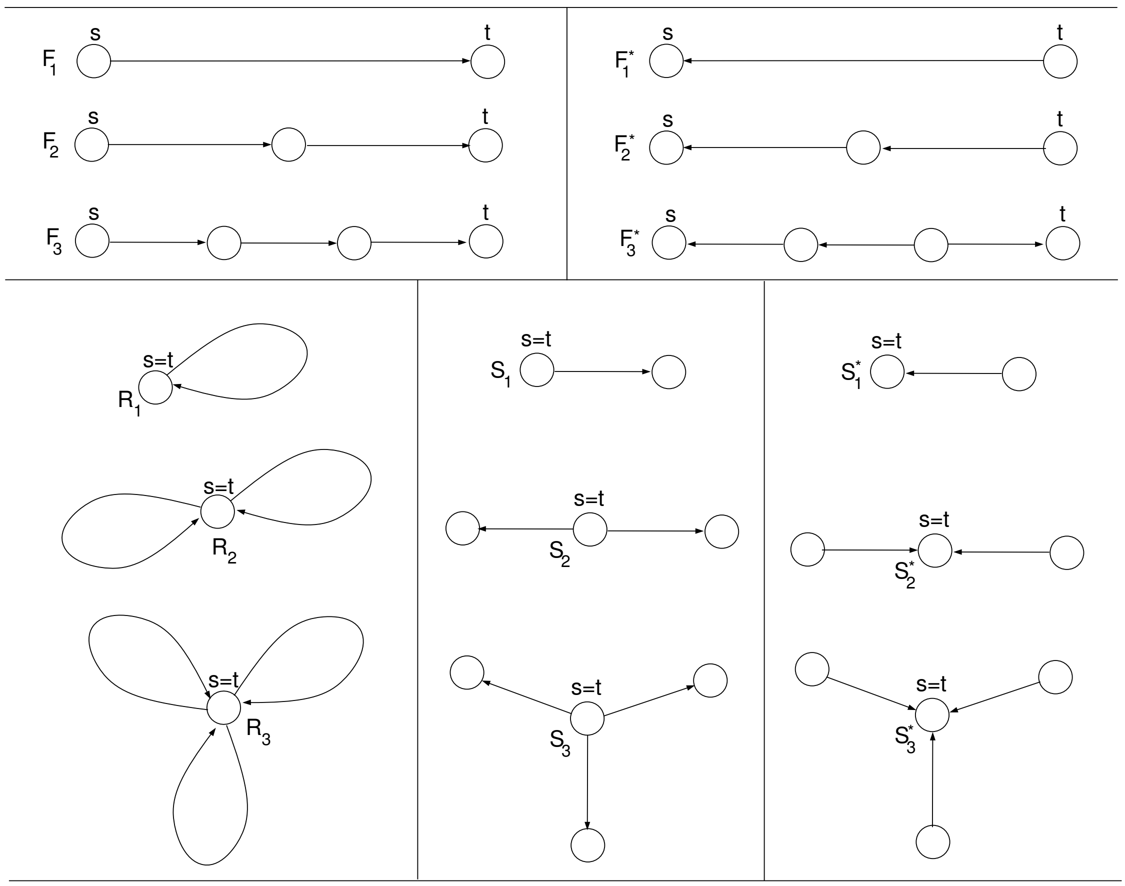

Figure 4: Some examples of special flow graphs: the graphical natural numbers F_1, F_2, F_3, the anti-paths F_1^*, F_2^*, F_3^*, the roses R_1, R_2, R_3, the stars S_1, S_2, S_3, and the anti-stars S_1^*, S_2^*, S_3^*.

1.1 Addition

We briefly review the operation of addition on Flow graphs as introduced in [1]. We begin by recalling the following ”Vertex gluing” operation on directed multigraphs:

Definition 1.12 (Vertex gluing of directed graphs)**.** *Given directed graphs G_1 and G_2, and vertices $u_1 \in V[G_1]$, $u_2 \in V[G_2]$, we define*

$$G_1 \underset{u_1 \approx u_2}{+} G_2 \ \stackrel{def}{=}\ (G_1 \sqcup G_2)/(u_1 \approx u_2)$$

to be the graph obtained by taking disjoint copies of G_1 and G_2 and identifying vertex u_1 in G_1 with vertex u_2 in G_2. Note the obvious and natural graph embeddings

$$\begin{aligned} \sigma^+_{u_1 \approx u_2} : G_1 &\hookrightarrow G_1 \underset{u_1 \approx u_2}{+} G_2 \\ \tau^+_{u_1 \approx u_2} : G_2 &\hookrightarrow G_1 \underset{u_1 \approx u_2}{+} G_2.. \end{aligned} \tag{1}$$

Now we can define addition of flow graphs:

Definition 1.13. *Given two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$, we define*

$$A+B \ \stackrel{\text{def}}{=}\ (G_A \underset{t_A \approx s_B}{+} G_B, s_A, t_B).$$

Since A and B are connected, it follows that $A+B$ is connected. This leads in a natural way to addition of flow graphs:

Definition 1.14. *Given two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$, we define*

$$A+B \ \stackrel{\text{def}}{=}\ (G_A \underset{t_A \approx s_B}{+} G_B, s_A, t_B).$$

An example of such an addition is shown in Figure 5.

Remark 1.15. *Note that if A is a flow graph with p_A vertices and q_A edges, and B is a flow graph with p_B vertices and q_B edges, then $A+B$ is a flow graph having $p_A + p_B - 1$ vertices and $q_A + q_B$ edges.*

Lemma 1.16. *Let m, n be natural numbers. Then $i(n+m) = i(n)+i(m)$.*

Lemma 1.17. *(See [1]) $0 \stackrel{\text{def}}{=} F_0$ is the unique two-sided identity with respect to $+$. That is, for all flow graphs $A, G \in \mathcal{F}$,*

$$A+G = A \ \Leftrightarrow\ G = 0 \ \Leftrightarrow\ G+A = A.$$

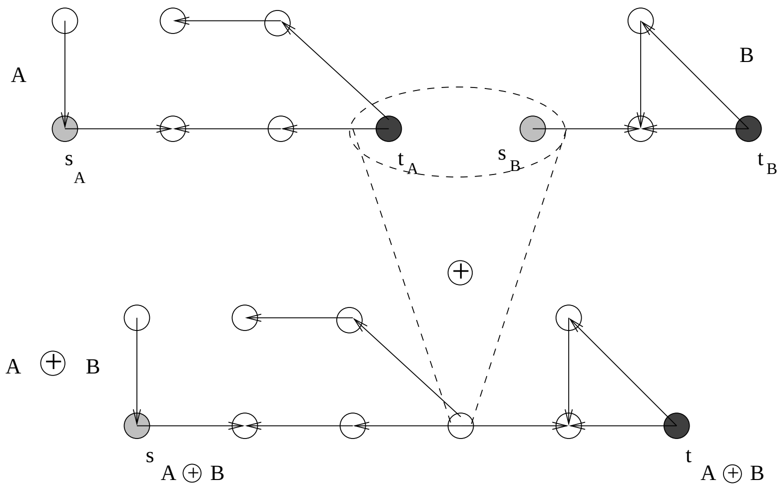

Figure 5: General addition of flow graphs.

Definition 1.18 (Scalar multiplication of flow graphs)**.** *Given a flow graph A, and a positive natural number k in $\mathbb{N}$, we define left scalar multiplication inductively as follows:*

$$\begin{aligned} 1A &= A \\ kA &= (k-1)A+A. \end{aligned}$$

Right scalar multiplication is defined analogously. However, as we will see, $+$ is associative, and so the two notions coincide. We shall subsequently consider only left scalar multiplication by integer scalars.

One can check that $1+R_1 \neq R_1+1$. Thus we obtain

Lemma 1.19. *The operation $+$ is not commutative.*

1.2 Multiplication

In this section, we recall from [1], multiplication in $\mathcal{F}$ which generalizes multiplication of natural numbers. In doing this, we must respect the fact that for each pair of natural numbers n_1, n_2, the following identity holds in $\mathcal{N}$:

$$\underbrace{n_2 + n_2 + \cdots + n_2}_{n_1 \text{ times}} = n_1 n_2 = \underbrace{n_1 + n_1 + \cdots + n_1}_{n_2 \text{ times}}.$$

So, in particular, the definition of in $\mathcal{F}$ must satisfy

$$n_1 F_{n_2} = F_{n_1} F_{n_2} = F_{n_1} n_2. \tag{2}$$

For example, the multiplication of graphical natural numbers F_3 and F_2 is illustrated in Figure 6.

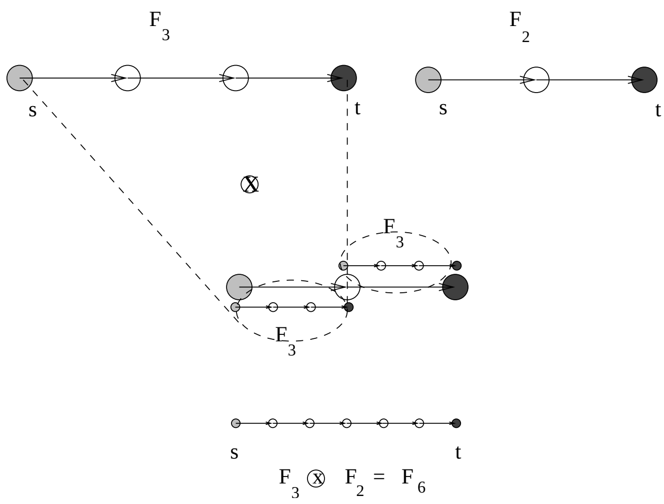

Figure 6: Standard multiplication of natural numbers in $\mathcal{F}$ (represented as flow graphs).

Definition 1.20. *Let $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$ be any two flow graphs. We define an equivalence relation $\sim_R$ on $V_A \times E_B$, as follows: Given vertices u_1, u_2 in V_A, and edges $e_1 = (v_1, w_1)$ and $e_2 = (v_2, w_2)$ in E_B, let $(u_1, (v_1, w_1)) \sim_R (u_2, (v_2, w_2))$ iff the following holds: whenever u_1 is the source (target) and u_2 is the source (target) then (respectively) the tail (head) of e_1 coincides with the tail (head) of e_2 in B. Then $\sim_R$ is an equivalence relation.*

We define the flow graph $AB = (G_{AB}, s_{AB}, t_{AB})$ as follows. Let $G_{AB} = (V_{AB}, E_{AB})$, where $V_{AB} = (V_A \times E_B)/\sim_R$ and $((u_1, e_1), (u_2, e_2)) \in E_{AB}$ if $(u_1, u_2) \in E_A$ and $e_1 = e_2$ in E_B. Define $s_{AB} = (s_A \times e)/\sim_R$ where $e = (s_B, w)$ for any $w \in V_B$ and $t_{AB} = (t_A \times e)/\sim_R$ where $e = (v, t_B)$ for any $v \in V_B$.

Since A and B are connected, it follows that AB is connected. An example of such a multiplication operation is shown in Figure 7.

Remark 1.21. *Let A be a flow graph with p_A vertices and q_A edges, and B be a flow graph having p_B vertices and q_B edges. Then AB has $q_A q_B$ edges. If A is either trivial or infinitesimal then AB has $1 + q_B(p_A - 1)$ vertices. If A is non-trivial and non-infinitesimal then AB has $p_B + q_B(p_A - 2)$ vertices.*

The next lemma follows immediately from Definitions 1.11 and 1.20.

Lemma 1.22. *Let m, n be natural numbers. Then $i(n \times m) = i(n)i(m)$.*

The next Lemma shows that $\mathcal{F}$ has no members which behave like zero divisors.

Lemma 1.23. *(See [1]) Given flow graphs G and H:*

$$GH = 0 \quad \Leftrightarrow \quad H = 0 \quad \textit{or} \quad G = 0. \tag{3}$$

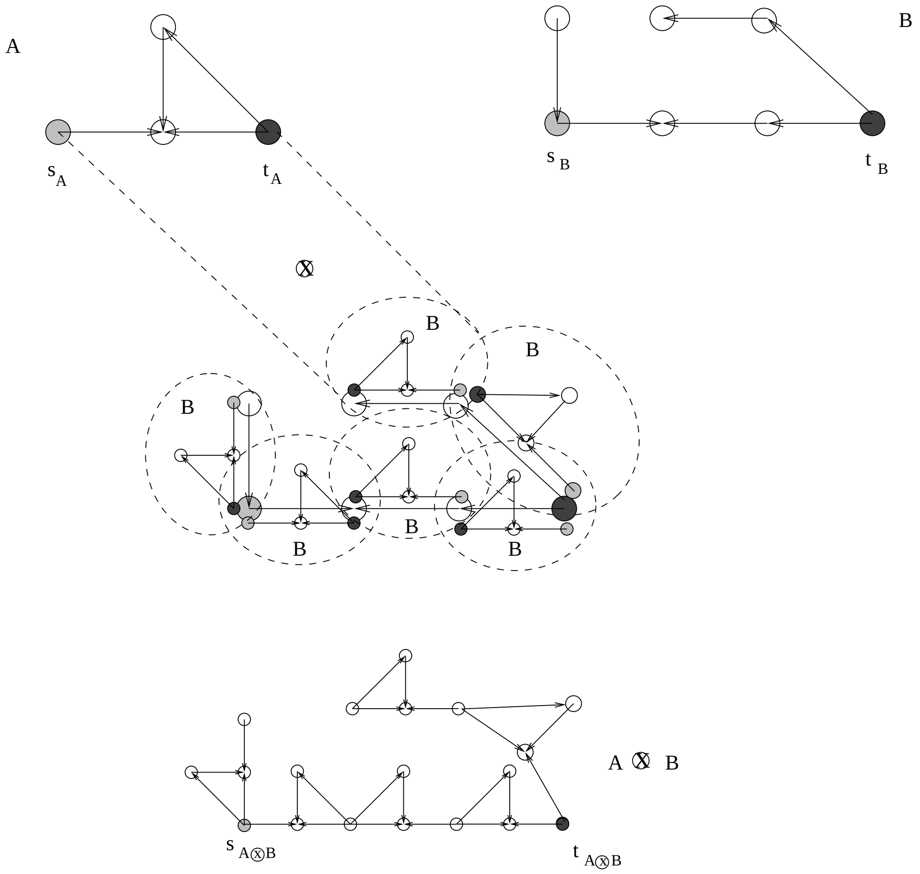

Figure 7: General multiplication of flow graphs.

One can check that $R_1 F_2 \neq F_2 R_1$. Thus we obtain

Lemma 1.24. *The operation is not commutative.*

Lemma 1.25. *(See [1]) If G and H are both non-trivial and non-infinitesimal, then*

$$\begin{aligned} HG = H \quad &\Leftrightarrow \quad G = 1, \\ GH = H \quad &\Leftrightarrow \quad G = 1. \end{aligned} \tag{4}$$

Definition 1.26 (Scalar exponentiation of flow graphs)**.** *Given a flow graph A, and a positive natural number k in $\mathbb{N}$, we define right-exponentiation inductively as follows:*

$$\begin{aligned} A^1 &= A \\ A^k &= A^{k-1} A. \end{aligned}$$

Left-exponentiation is defined analogously. However, as we will see shortly, is associative, and so the two notions coincide. We shall subsequently consider only right-exponentiation by integer scalars.

1.3 Infinitesimals

The following observations motivate our choice of the term *infinitesimal* for flow graphs whose source and target vertices coincide.

Proposition 1.27. *(See [1]) Let B and C be non-trivial flow graphs. Then $B+C$ is infinitesimal, if and only if both B and C are infinitesimal.*

The next Proposition shows that with respect to multiplication, the set of infinitesimals behaves, in some sense, like a prime ideal inside F.

Proposition 1.28. *(See [1]) Let G and H be non-trivial flow graphs, then GH and HG are infinitesimal if and only if at least one of the two factors is infinitesimal.*

The reader may wish to compare the above Proposition with assertion (3) of Lemma 1.23 which showed that $\{0\}$ also behaves, in some sense, like a prime ideal inside F.

2 Results

We begin by considering properties of $+$ in Section 2.1. We prove that every flow graph is canonically decomposable as a sum of irreducibles and using this canonical decomposition, we deduce left and right cancellation laws for $+$.

2.1 Additive Properties

We now define the concept of $+$irreducibile.

Definition 2.1 ($+$Irreducible)**.** *A flow graph A is called* $+$reducible *if there exist non-trivial flow graphs B, C, such that $A = B+C$. Otherwise, A is called $+$irreducible.*

We would like to obtain a graph-theoretic characterization of $+$irreducibility. Towards this, we introduce the following graph operation called $*$**Deletion.**

Definition 2.2 ($*$-Deletion of a vertex)**.** *Let $G = (V, E)$ be directed connected multigraph and let w be a vertex in V. We define $G\backslash_* w$ to be the graph with components $C_1^*, \ldots, C_{k(w)}^*$, obtained by the following process:*

(1) For each non-loop directed edge $e = (w, u)$ incident to w, we introduce a vertex v_e, and replace the edge e with the directed edge (w, v_e) and a loop (v_e, u).

(2) *For each non-loop directed edge $e = (u, w)$ incident to w, we introduce a vertex v_e, and replace the edge e with the directed edges (u, v_e) and (v_e, w).*

(3) *For each directed loop $e = (w, w)$ we introduce a vertex v_e, and replace the directed loop e with directed edges (w, v_e) and a loop (v_e, v_e).*

(4) *Delete w and all incident edges. Denote the collection of connected components after w is deleted, as $C_1, \ldots, C_{k(w)}$.*

(5) *Let P be the set of all vertices added in steps (1-3). Two vertices v_{e_1} and v_{e_2} in P are said to be* related by R_w *if they lie in the same component C_i for some i in $\{1, \ldots, k(w)\}$. Clearly this defines an equivalence relation on P. In step 5, we identify all vertices that are R_w-related.*

The resulting collection of connected components is denoted $C_1^*, \ldots, C_{k(w)}^*$. The set of vertices obtained in step 5 is denoted P/R_w, and is indexed $\{w_1, w_2, \ldots, w_{k(w)}\}$ so that w_i lies in C_i^* for $i = 1, \ldots, k(w)$.

Two examples of $*$-deletion are depicted in *the first four stages* of the process shown in Figures 8 and 9 on pages 11 and 15 respectively.

Lemma 2.3. *Let w be a vertex in a directed connected multigraph $G = (V, E)$. Then $G \backslash_* w = G$ if and only if w is not a cut vertex*[4] *in G.*

Proof. If w is not a cut vertex then $k(w) = 1$ and following the construction, w_1 replaces w in $G \backslash_* w = G$. If w is a cut vertex then $k(w) > 1$. Since G is connected and $G \backslash_* w$ is not, it follows that $G \backslash_* w \neq G$. □

We now introduce the flow graph analogue of the classical notion of a cut vertex.

Definition 2.4 (Splitting vertex of a flow graph)**.** *We say that w is a* splitting vertex *for a flow graph $A = (G_A, s_A, t_A)$, if one of the following holds:*

(I) *$w \neq s_A$, $w \neq t_A$, and s_A and t_A lie in distinct components of $G \backslash_* w$.*

(II) *$w = s_A$ or $w = t_A$ (or both), and $G \backslash_* w$ contains at least two non-trivial components.*

Remark 2.5. *In light of Lemma 2.3 and Definition 2.4, we observe that if w is a splitting vertex in a flow graph $A = (G_A, s_A, t_A)$ then w is a cut vertex in the graph G_A. The converse is false, however, since not every cut vertex in G_A is a splitting vertex in A. A concrete example is seen in the flow graph A of Figure 5 on page 6, where A contains two cut vertices for G_A, both of which are at distance 1 from t_A but only one of which is a splitting vertex for A.*

[4] By cut vertex, we mean a vertex whose deletion results in at least two non-trivial components.

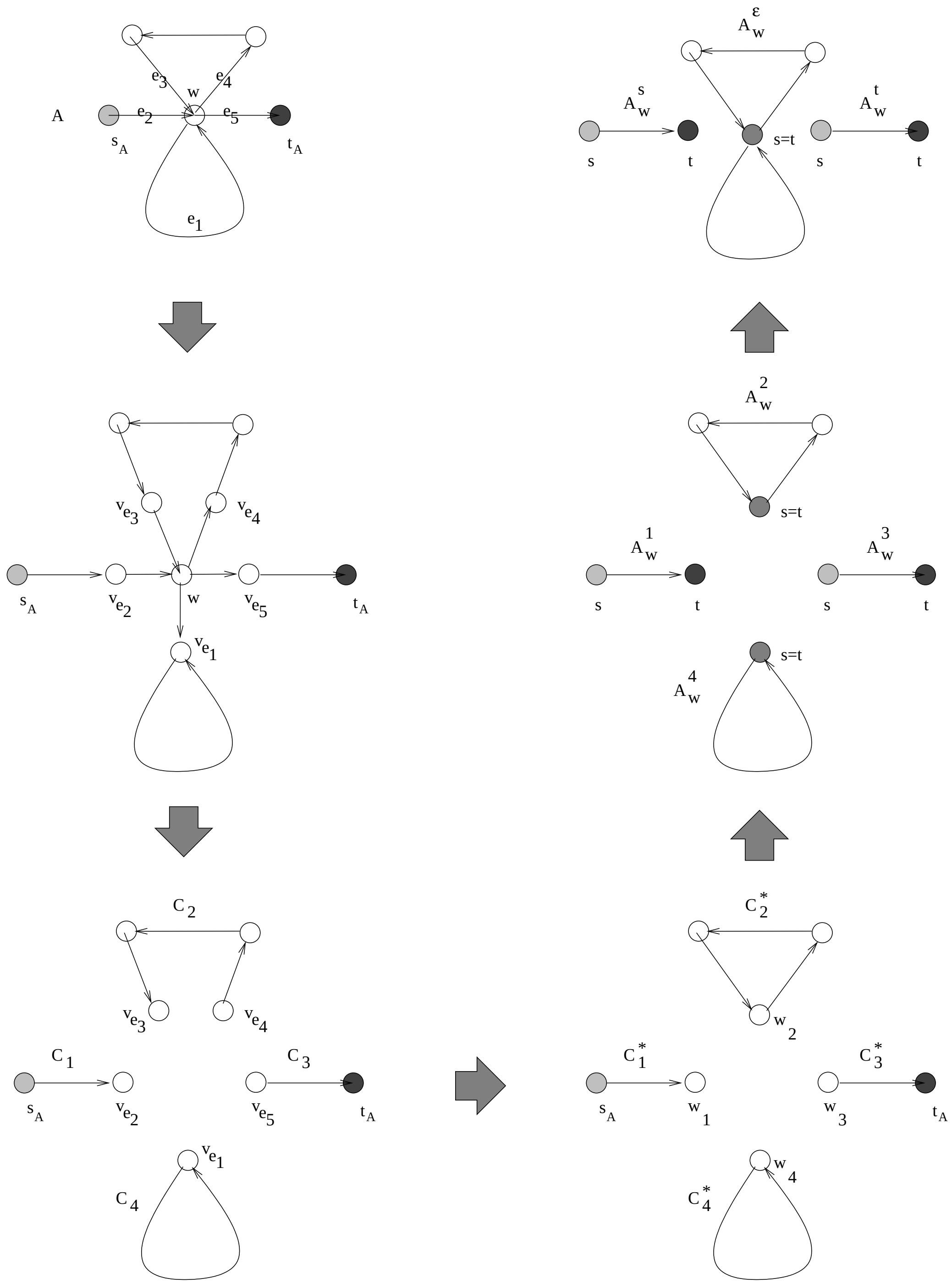

Figure 8: The w-splitting of a non-infinitesimal flow graph A resulting in a non-trivial splitting.

We introduce additional structure on the components produced upon $*$-deletion of a splitting vertex from a flow graph.

Definition 2.6 (w-Splitting of a flow graph)**.** *Let A be a flow graph and w a splitting vertex for A. Considering the components of $A\backslash_* w$, denoted $C_1^*, \ldots, C_{k(w)}^*$, we define flow graphs $A_w^1, \ldots, A_w^{k(w)}$ as follows. For each $i = 1, \ldots k(w)$:*

$$A_w^i = \begin{cases} (C_i^*, s_A, w_i) & \textit{if } s_A \in V[C_i^*] \\ (C_i^*, w_i, t_A) & \textit{if } t_A \in V[C_i^*] \\ (C_i^*, w_i, w_i) & \textit{otherwise.} \end{cases} \tag{5}$$

We distinguish (at most) two members of the set $\{A_w^1, \ldots, A_w^{k(w)}\}$

$$A_w^s = \begin{cases} A_w^i & \textit{if } s_A \in V[C_i^*] \\ 0 & \textit{if } \forall i \in \{1, \ldots, k(w)\}, s_A \notin V[C_i^*] \end{cases} \tag{6}$$

$$A_w^t = \begin{cases} A_w^i & \textit{if } t_A \in V[C_i^*] \\ 0 & \textit{if } \forall i \in \{1, \ldots, k(w)\}, t_A \notin V[C_i^*] \end{cases} \tag{7}$$

The remaining members of the set $\{A_w^1, \ldots, A_w^{k(w)}\}$ are collected as follows. Let

$$\epsilon_w = \{i \mid s_A, t_A \notin V[C_i^*]\} \tag{8}$$

$$A_w^\epsilon = \begin{cases} +_{i \in \epsilon_w} A_w^i & \epsilon_w \neq \emptyset. \\ 0 & \textit{otherwise.} \end{cases} \tag{9}$$

The triple of flow graphs $(A_w^s, A_w^\epsilon, A_w^t)$ is called the w-splitting of A.

The *final two stages* in Figures 8 and 9 depict the transformation of $A\backslash_* w$ into the w-splitting of A.

Lemma 2.7. *Let A be a flow graph and w a splitting vertex for A.*

$$A_w^s = 0 \;\Rightarrow\; s_A = w$$
$$A_w^t = 0 \;\Rightarrow\; t_A = w$$

Proof. If $A_w^s = 0$, then by expression (6), $s_A \notin V[C_i^*]$ for all i in $\{1, \ldots, k(w)\}$, so w must be s_A, since this is the only vertex that is eliminated in the process of $*$-deletion. A similar argument holds for the case when $A_w^t = 0$ showing that $t_A = w$. □

The next Remark follows from the definition of the splitting process.

Remark 2.8. *If a flow graph $A = (G_A, s_A, t_A)$ has a splitting vertex w, then*

$$\begin{aligned} A &= +_{i=1,\ldots,k(w)} A_w^i \\ &= A_w^s + (+_{i\in\epsilon_w} A^i) + A_w^t \\ &= A_w^s + A_w^\epsilon + A_w^t. \end{aligned}$$

Lemma 2.9. *For each $i = 1, \ldots, k(w)$, the flow graph A_w^i is non-trivial.*

Proof. In steps $1-4$ in the construction of $G\backslash_* w$ (see Definition 2.2), no component C_i can have fewer than 1 edge. Since C_i^* is obtained by identifying vertices in C_i, no C_i^* can have fewer than 1 edge. □

Definition 2.10. *A splitting of the form $(0, A, 0)$ is called a* trivial splitting.

Remark 2.11. *If A is irreducible, then A has a trivial splitting. But conversely is false as the next example shows.*

Example 2.12. *A flow graph may contain splitting vertex w and yet only possess trivial w-splittings. Consider, for example, the unique vertex w in the flow graph*

$$A = S_1 + R_1 + S_1^* + (F_3 R_1)$$

shown in Figure 9. Clearly, w is a splitting vertex, but the w-splitting of R_2 is $(0, A, 0)$.

The next lemma shows that the phenomenon exhibited by the previous example is present in all infinitesimal flow graphs which possess a splitting vertex.

Lemma 2.13. *Let A be an infinitesimal flow graph with splitting vertex w. Then*

(i) *w necessarily coincides with $s_A = t_A$, and $|\epsilon_w| \geqslant 2$.*

(ii) *the w-splitting of A is the trivial splitting $(0, A_w^\epsilon, 0)$, where $A_w^\epsilon = A$.*

Proof. If A is infinitesimal $s_A = t_A$, so w must satisfy condition (II) of Definition 2.4. Thus, $w = s_A = t_A$ and $G\backslash_* w$ has at least two non-trivial components, completing the proof of assertion (i). Since step 4 in the construction of $G\backslash_* w$ (see Definition 2.2) requires deleting w, no component C_i^* contains $w = s_A = t_A$. It follows from expressions (6) and (7) in Definition 2.6 that $A_w^s = A_w^t = 0$. By Remark 2.8, $A = 0 + A_w^\epsilon + 0$, and hence $A_w^\epsilon = A$. This proves the second assertion. □

Of course, Lemma 2.13 does not apply to every infinitesimal flow graph, since not every infinitesimal flow graph has a splitting vertex. For example, $F_3 R_1$ has no splitting vertex. We shall see later that if a *non-infinitesimal* flow graph has a splitting vertex then it must possess a non-trivial splitting.

Lemma 2.14. *Let A be an arbitrary flow graph with splitting vertex w. Then*

- *(i) For each $i \in \epsilon_w$, the flow graphs A_w^i is non-trivial, infinitesimal and contains no splitting vertices.*
- *(ii) A_w^ϵ is infinitesimal.*
- *(iii) A_w^s and A_w^t are either trivial or not infinitesimal.*
- *(iv) No two adjacent elements in the triple $(A_w^s, A_w^\epsilon, A_w^t)$ are trivial.*

Proof. (i) If $\epsilon_w = \emptyset$ the statement is vacuously true, so we may assume $|\epsilon_w| \geqslant 1$. For each $i \in \epsilon_w$ (see expression (8) in Definition 2.6) A_w^i is non-trivial (by Remark 2.9) and infinitesimal (by expression (5) of Definition 2.6). Now if A_w^i (for $i \in \epsilon_w$) contains a splitting vertex, then by part (i) of Lemma 2.13, the splitting vertex must be $w_i = s_{A_w^i} = t_{A_w^i}$. But w_i (in C_i^*) is the identification of a set of vertices (in the *connected graph* C_i). Hence, w_i is not a cut vertex in C_i^*, and therefore A_w^i contains no splitting vertex.

(ii) By expression (9) of Definition 2.6, A_w^e is either trivial, or it is the sum of flow graphs that are infinitesimal, by part (i). Since the sum of infinitesimals is infinitesimal, the result follows.

(iii) If A_w^s trivial, then we are done. If A_w^s is non-trivial, then $A_w^s = A_w^i$ for some i in $\{1, \ldots, k(w)\}$, with $s_A \in V[C_i^*]$. Since $s_A \neq w_i \in V[C_i^*]$ and w is a splitting vertex, Definition 2.4 implies $s_A \neq t_A$. Thus A is not infinitesimal. The proof for A_w^t is analogous.

(iv) Since w is a splitting vertex of A, it satisfies condition (I) or (II) of Definition 2.4. If w satisfies (I), then s_A and t_A lie in different components of $G\backslash_* w$ and by Remark 2.9, A_w^s and A_w^t are both non-trivial. On the other hand, if w satisfies (II) then $w = s_A$, or $w = t_A$, or both—hence either $A_w^t = 0$, or $A_w^s = 0$, or both. But since $G\backslash_* w$ contains at least two non-trivial components, it follows that $|\epsilon_w| \geqslant 1$, and hence $A_w^\epsilon \neq 0$. ☐

The next Lemma provides a graph-theoretic characterization of $+$-irreducibility.

Lemma 2.15 ($+$Irreducibility Lemma)**.** *A flow graph $A = (G_A, s_A, t_A)$ is $+$reducible if and only if A has a splitting vertex.*

Proof. ($\Rightarrow$) Suppose $A = B{+}C$. Then, take $w = \sigma^+_{t_B \approx s_C}(t_B) = \tau^+_{t_B \approx s_C}(s_C)$ to be the vertex in G_A obtained in identifying t_B with s_C.

If B is not infinitesimal, then $w = t_B \neq s_B = s_A$, so it follows that s_A is in some component A_w^i (for i in $\{1, 2, 3, \cdots, k(w)\}$); hence $A_w^s \neq 0$. By a similar argument, if C is not infinitesimal then t_A is in some component A_w^i (for i in $\{1, 2, 3, \cdots, k(w)\}$); hence $A_w^t \neq 0$. If B and C are both non-infinitesimal, then $w \neq s_A$, $w \neq t_A$, and s_A and t_A lie in distinct components of $G\backslash_* w$. So w is a splitting vertex by case (I) of Definition 2.4.

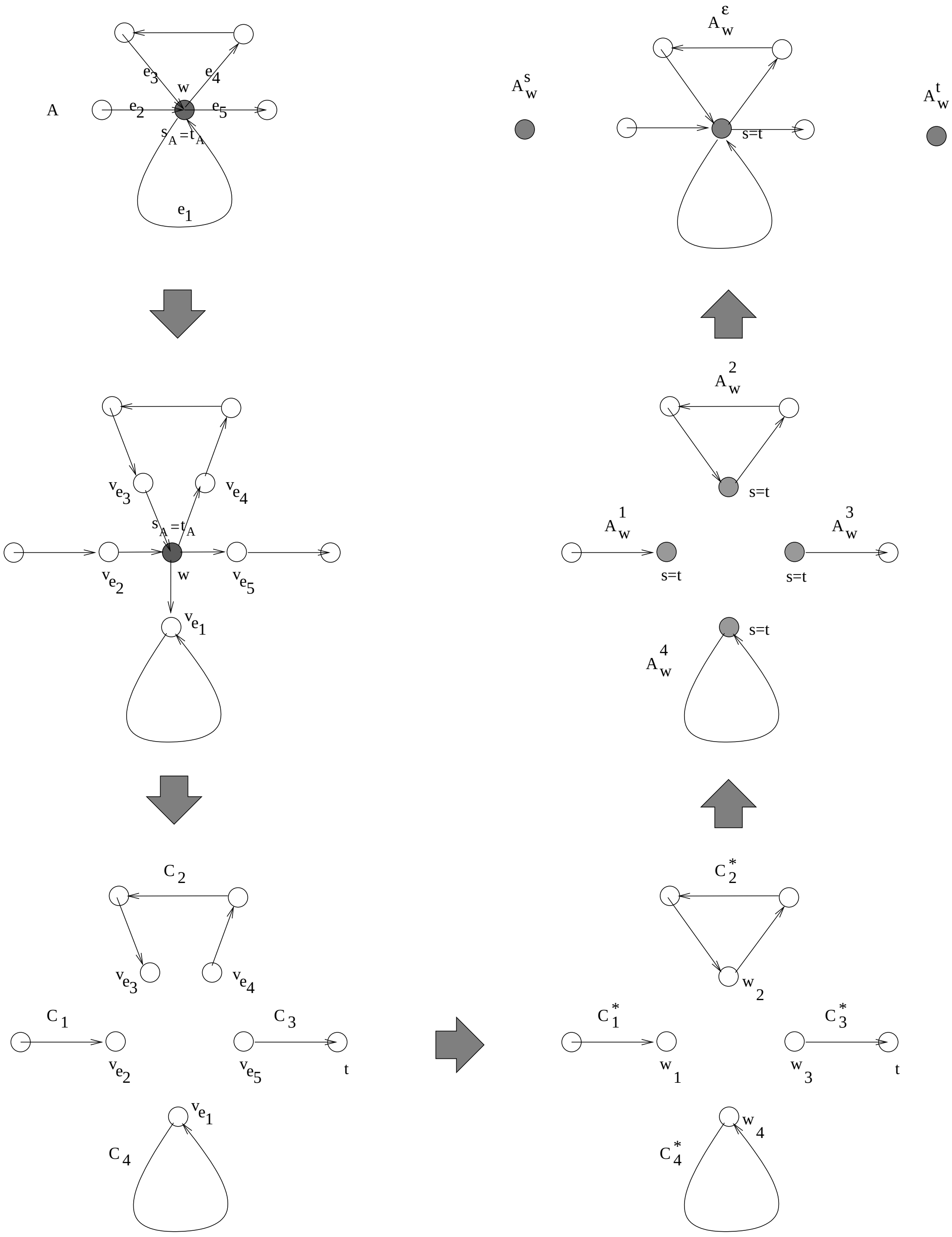

Figure 9: The w-splitting of an infinitesimal flow graph A, yielding a trivial splitting.

If B is infinitesimal, then since $w = t_B = s_B = s_A$, it follows that s_A is not in any component A_w^i (for i in $\{1, 2, 3, \cdots, k(w)\}$); hence $A_w^s = 0$ and by part (iv) of Lemma 2.14, $A_w^\epsilon \neq 0$. By a similar argument, if C is infinitesimal then $A_w^t = 0$ and $A_w^\epsilon \neq 0$. If B is infinitesimal and C is non-infinitesimal, then $A_w^\epsilon \neq 0$ and $A_w^t \neq 0$ are two non-trivial components in $G\backslash_* w$. If B is non-infinitesimal and C is infinitesimal, then $A_w^s \neq 0$ and $A_w^\epsilon \neq 0$ are two non-trivial components in $G\backslash_* w$. If both B and C are infinitesimal then A is infinitesimal also, so by part (i) of Lemma 2.13, $|\epsilon_w| \geqslant 2$, and $\{A_w^i \mid i \in \epsilon_w\}$ contributes at least two non-trivial components to $G\backslash_* w$. In all these cases, w is a splitting vertex by case (II) of Definition 2.4.

($\Leftarrow$) Suppose A has a splitting vertex w. Then $A = A_w^s + A_w^\epsilon + A_w^t$. If w satisfies case (I) of Definition 2.4, then taking $B = A_w^s$ and $C = A_w^\epsilon + A_w^t$, we can express A as the sum of two flow graphs which are each non-trivial (by part (iv) of Lemma 2.14). Thus A is $+reducible$.

If w satisfies case (II) of Definition 2.4, then either A_w^s is trivial or A_w^t or both. If only A_w^s is trivial, then taking $B = A_w^\epsilon$ and $C = A_w^t$, we can express A as the sum of two flow graphs which are each non-trivial (by part (iv) of Lemma 2.14). If only A_w^t is trivial, then taking $B = A_w^s$ and $C = A_w^\epsilon$, we can express A as the sum of two flow graphs which are each non-trivial (by part (iv) of Lemma 2.14). If both A_w^s and A_w^t are trivial, then A is infinitesimal and by part (i) of Lemma 2.13, $|\epsilon_w| \geqslant 2$. We can therefore partition $\epsilon_w = \epsilon_w^1 \cup \epsilon_w^2$, taking $B = +_{i \in \epsilon_w^1} A_w^i$ and $C = +_{i \in \epsilon_w^1} A_w^i$, thereby expressing A as the sum of two flow graphs which are each non-trivial (by part (i) of Lemma 2.14). In all cases, we have shown A is $+reducible$. $\square$

Definition 2.16 (Splitting vertex ranking)**.** *Given flow graph* $A = (G_A, s_A, t_A)$*, let* $\chi(A) \subset V[G_A]$ *be the set of all splitting vertices for* A*. We define the* s-ranking *and* t-ranking *functions* $r_s^A, r_t^A : \chi(A) \to \mathbb{N}$ *as follows:*

$$\begin{aligned} r_s^A(w) &= |V[G_{A_w^s}] \cap \chi(A)|, \\ r_t^A(w) &= |V[G_{A_w^t}] \cap \chi(A)|. \end{aligned}$$

When it is clear from the context, we denote $r_s(w) = r_s^A(w)$ *and* $r_t(w) = r_t^A(w)$*.*

Lemma 2.17. *Let* $w \in \chi(A)$ *be a splitting vertex for flow graph* $A = (G_A, s_A, t_A)$*. Then for all* $u \in V[G_{A_w^s}] \cap \chi(A)$*:*

$$\begin{aligned} r_s(u) &< r_s(w), \\ r_t(u) &> r_t(w); \end{aligned}$$

and for all $u \in V[G_{A_w^t}] \cap \chi(A)$*:*

$$\begin{aligned} r_s(u) &> r_s(w), \\ r_t(u) &< r_t(w). \end{aligned}$$

Proof. First, note that for any u in $(V[G_{A_w^s})] \cup V[G_{A_w^t}]) \cap \chi(A)$

$$r_s(u) + r_t(u) + 1 = |\chi(A)| \tag{10}$$

Now if $u \in V[G_{A^s_w}] \cap \chi(A)$, then since $w \in (V[G_{A^t_u}] \backslash V[G_{A^t_w}]) \cap \chi(A)$, it follows that $V[G_{A^t_w}] \subsetneq V[G_{A^t_u}]$. But then $r_t(w) < r_t(u)$. By expression 10 above $r_s(w) > r_s(u)$. The proof for the case when $u \in V[G_{A^t_w}] \cap \chi(A)$ is analogous. □

Lemma 2.18. *Given a flow graph $A = (G_A, s_A, t_A)$, for each $i = 0, 1, \ldots, |\chi(A)| - 1$ there is a unique vertex v_i in $\chi(A)$ with the property that $r_s(v_i) = i$.*

Proof. Given two distinct vertices v, v' in $\chi(A)$, either $v \in V[G_{A^s_{v'}}]$ or $v \in V[G_{A^t_{v'}}]$. In the latter circumstance, $v' \in V[G_{A^s_v}]$ so it follows from Lemma 2.17 that $r_s(v) \neq r_s(v')$. Base case: $i = 0$. Let w_0 be any vertex in $\chi(A)$. If $r_s(w_0) > 0$, then $V[G_{A^s_w}] \cap \chi(A)$ is not empty. So let w_1 be any vertex in $V[G_{A^s_w}] \cap \chi(A)$. By Lemma 2.17, $r_s(w_1) < r_s(w_0)$. Repeating in this fashion, after finitely many steps $w_0 \rightsquigarrow w_1 \rightsquigarrow \ldots$ we find some vertex v_0 for which $r_s(v_0) = 0$. Inductive step $i+1$: Let v_i be the unique vertex in $\chi(A)$ having $r_s(v_i) = i$. Define v_{i+1} to be the vertex in $V[G_{A^t_{v_i}}] \cap \chi(A)$ for whose s-rank is minimal. Since

$$V[G_{A^s_{v_{i+1}}}] \cap \chi(A) \quad = \quad \Big(V[G_{A^s_{v_i}}] \cap \chi(A)\Big) \cup \{v_i\},$$

it follows that $r_s(v_{i+1}) = r_s(v_i) + 1 = i + 1$, hence the result. □

Definition 2.19 (Canonical $+$-decomposition)**.** *Let $A = (G_A, s_A, t_A)$ be a flow graph. If A is infinitesimal or if $\chi(A) = 0$, then the canonical $+$-decomposition of A is defined to be the formal sum $0{+}A{+}0$.*

Otherwise, let $\chi(A) = \{v_0, v_1, \ldots, v_{|\chi(A)|-1}\}$ be the non-empty set of splitting vertices of A, ordered according to the indexing scheme postulated in Lemma 2.18. Define $A^{(0)} = A^s_{v_0}$, $A^{\epsilon(0)} = A^{\epsilon}_{v_0}$, $\bar{A}^{(0)} = A^t_{v_0}$, and then for each $i = 1, 2, \ldots, |\chi(A)| - 1$, put

$$\begin{aligned} A^{(i)} &= (\bar{A}^{(i-1)})^s_{v_i}, \\ A^{\epsilon(i)} &= (\bar{A}^{(i-1)})^{\epsilon}_{v_i}, \\ \bar{A}^{(i)} &= (\bar{A}^{(i-1)})^t_{v_i}. \end{aligned}$$

We shall denote $\bar{A}^{(|\chi(A)|-1)}$ as $A^{(|\chi(A)|)}$. The canonical $+$-decomposition of A *is defined to be the* formal *summation:*

$$\langle A \rangle \stackrel{\text{def}}{=} A^{(0)}{+}A^{\epsilon(0)}{+}A^{(1)}{+}A^{\epsilon(1)} \ldots {+}A^{(|\chi(A)|-1)}{+}A^{\epsilon(|\chi(A)|-1)}{+}A^{(|\chi(A)|)}.$$

Note that Lemma 2.18 and effectiveness of the definition above together guarantee the uniqueness of the decomposition.

Definition 2.20 (Alternating sum)**.** *A sum $A_0{+}A_1{+}\cdots{+}A_{2k}$ ($k \in \mathbb{N}$) is called an* alternating sum *of weight k if*

$$A_i \quad \textit{is} \quad \begin{cases} \textit{infinitesimal} & \textit{if } i \textit{ is odd,} \\ \textit{non-trivial, non-infinitesimal, and } +\textit{-irreducible} & \textit{if } 0 < i < 2k, i \textit{ is even,} \\ \textit{non-infinitesimal and } +\textit{-irreducible, or trivial} & \textit{if } i = 0 \textit{ or } i = 2k. \end{cases}$$

no two adjacent elements A_i, A_{i+1} ($i = 0, \ldots, 2k-1$) are trivial.

Remark 2.21. *It is easy to see that given a flow graph A expressed as an alternating sum:*

$$A = A_0 + A_1 + \cdots + A_{2m}$$

the set of its splitting vertices $\chi(A) = \chi(A_0 + A_1 + \cdots + A_{2m})$ is precisely the image of $\{s_{A_i} \mid i \text{ odd}\}$ under the suitable embeddings of $A_i \hookrightarrow A_0 + A_1 + \cdots + A_{2m}$.

Proposition 2.22 (Correctness of the $+$-decomposition)**.** *Let $A = (G_A, s_A, t_A)$ be a flow graph. Then $\langle A \rangle$ is an alternating sum of weight $|\chi(A)|$ which satisfies $\langle A \rangle = A$.*

Proof. If $|\chi(A)| = 0$, then $\langle A \rangle = A$ trivially. Suppose $|\chi(A)| = n > 1$. Let w be the unique vertex for which $r_s^A(w) = 0$. Formally, $\langle A \rangle = A_w^s + A_w^\epsilon + \langle A_w^t \rangle$. Since $|\chi(A_w^t)| = n - 1$, and by inductive hypothesis, $\langle A_w^t \rangle = A_w^t$. Appealing to Observation 2.8, $A = A_w^s + A_w^\epsilon + A_w^t = \langle A \rangle$. □

Given a flow graph A, we see that $\langle A \rangle$ is an alternating sum. The next proposition shows that it is canonical, and that upto isomorphism, there is only one alternating sum which evaluates to A, namely $\langle A \rangle$.

Proposition 2.23 (Component-wise decomposition of isomorphisms under $+$)**.** *Suppose A and B are two isomorphic flow graphs, expressed as alternating sums:*

$$\begin{aligned} A &= A_0 + A_1 + \cdots + A_{2m}, \\ B &= B_0 + B_1 + \cdots + B_{2n} \end{aligned}$$

Then $m = n$ and every isomorphism $\phi : A \to B$ satisfies $\phi(A_i) = B_i$ (for $i = 1, \ldots m$).

Proof. Clearly, $\phi|_{\chi(A)}$ a bijection from $\chi(A)$ to $\chi(B)$. By Remark 2.21, for each i in $\{0, \ldots, |\chi(A)| - 1\}$, there exists some j in $\mathbb{N}$, such that $\phi(s_{A_{2i+1}}) = s_{B_{2j+1}}$. Take u_i to be the image of $s_{A_{2i+1}}$ in A under the natural embedding $A_{2i+1} \hookrightarrow A$, and v_j to be the image of $s_{B_{2j+1}}$ in B under the natural embedding $B_{2j+1} \hookrightarrow B$. Then $\phi(u_i) = v_j$ implies ϕ maps $\chi(A_{u_i}^s)$, having size i, bijectively onto $\chi(B_{v_j}^s)$, having size j. It follows that $i = j$. Since $\phi(u_i) = v_i$, the u_i splitting of A equals the v_i splitting of B, i.e.

$$\begin{aligned} \phi(A_{u_i}^s) &= B_{v_i}^s \\ \phi(A_{u_i}^\epsilon) &= B_{v_i}^\epsilon \\ \phi(A_{u_i}^t) &= B_{v_i}^t. \end{aligned}$$

In the case when $i = m - 1$, $u_{m-1} = s_{A_{2(m-1)+1}}$ and $v_{m-1} = s_{B_{2(m-1)+1}}$. Since $\chi(A_{u_{m-1}}^t)$ is empty, $\chi(B_{v_{m-1}}^t)$ must also be empty. Hence $2(m-1) + 1 = 2n - 1$, and therefore $m = n$.

Now if $|\chi(A)| = 0$ the theorem is trivially true. Suppose $|\chi(A)| = m > 0$. Then $\phi(u_{m-1}) = v_{m-1}$ and so $\phi(A^s_{u_{m-1}}) = B^s_{v_{m-1}}$. Since $|\chi(A^s_{u_{m-1}})| = m-1$ the inductive hypothesis applies and ϕ maps $A_0+A_1+\cdots+A_{2(m-1)}$ componentwise onto $B_0+B_1+\cdots+B_{2(m-1)}$. Since A_{m-1}, B_{m-1} are infinitesimal or trivial, and A_m, B_m are non-infinitesimal or trivial (but not both can be trivial, it follows that ϕ maps A_{m-1} onto B_{m-1} and A_m onto B_m. $\square$

3 Conclusions and Future Work

As we have seen, some theorems that are true in $\mathcal{N}$ continue to hold in $\mathcal{F}$, while others fail. Our future research program will proceed to explore further properties of flow graphs.

Some questions we are presently considering are listed below.

1. Characterize -commuting pairs, i.e. under what conditions on flow graphs A and B does $AB = BA$?
2. *Graph Prime Factorization Conjecture.* Every flow graph is uniquely expressible (up to some well-defined reordering) as the product of prime flow graphs.

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